

# A result on convergence of sequences of iterations, with applications to best-response dynamics

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## Abstract

The result which says that the sequence of iterations  $x^{k+1} = f(x^k)$  converges if  $f : [0, 1] \rightarrow [0, 1]$  is an increasing function, has numerous applications in elementary economic analysis. I generalize this simple result to some mappings  $f : S \subset [0, 1]^n \rightarrow S$ . The applications include, but are not limited to, the convergence of best-response dynamics in the general version Crawford and Sobel (1982) model.

## 1 Introduction

Studying sequences of iterations  $(x^k)_{k=0}^\infty$  defined inductively by  $x^{k+1} = f(x^k)$  for some mapping  $f$  is an important tool of numerical analysis, commonly used in mathematics, economics, computer science, and presumably numerous other disciplines. The convergence of sequences of iterations is often an important issue, and a number of famous convergence results have been established. The best-known examples include the convergence of all  $(x^k)_{k=0}^\infty$  to the unique fixed point for contraction mappings (Banach, 1922), and the convergence of sequences  $(x^k)_{k=0}^\infty$  to the highest and lowest fixed points for order-preserving functions defined on complete lattices (Tarski, 1955).

In this note, I am concerned with a specific, but important result of this kind, which says that the sequence  $(x^k)_{k=0}^\infty$  converges if  $f : [0, 1] \rightarrow [0, 1]$  is an increasing function, and converges to a fixed point of  $f$  if  $f$  is in addition continuous. This fact has numerous applications in elementary economic analysis. (I give later an example of its less straightforward application.) More specifically, I aim to generalize this simple result to mappings  $f : [0, 1]^n \rightarrow [0, 1]^n$ . Assuming a “lattice monotonicity”<sup>1</sup> is insufficient for obtaining such a generalization, and assuming “coordinate-by-coordinate monotonicity”<sup>2</sup> only reiterates the result for functions from  $[0, 1]$  into itself, and as such, it cannot have interesting new applications.

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<sup>1</sup>By lattice monotonicity of  $f = (f_1, \dots, f_n)$ , I understand that if  $x_1 \leq y_1, \dots, x_n \leq y_n$ , then  $f_1(x) \leq f_1(y), \dots, f_n(x) \leq f_n(y)$ , where  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ .

<sup>2</sup>By coordinate-by-coordinate monotonicity, I mean that for every  $i = 1, \dots, n$ , if  $x_i \leq y_i$ , then  $f_i(x) \leq f_i(y)$ .

I provide fairly simple conditions on mapping  $f$  which guarantee the convergence of  $(x^k)_{k=0}^\infty$ , and such that the result obtained in this manner has new, and nontrivial economic applications. My conditions will guarantee that the sequence  $(x^k)_{k=0}^\infty$  is monotonic, except a finite (but possibly very high) number of iterations  $k$ . The convergence is established by introducing an index which counts the number of coordinates  $i$  such that  $x_i^{k-1} > x_i^k$  and  $x_i^k < x_i^{k+1}$  or  $x_i^{k-1} < x_i^k$  and  $x_i^k > x_i^{k+1}$ .<sup>3</sup> The conditions imposed on mapping  $f$  guarantee that this index is equal to zero after a finite number of iterations, which means that for all  $i$ , the sequence  $(x_i^k)_{k=1}^\infty$  is monotonic, beginning from some  $k$ .

The applications include, but are not limited to the convergence of the best-response dynamics in the celebrated Crawford and Sobel (1982) model. Some partial convergence results in this setting were established earlier by Gordon (2011), and Gordon et al. (2019). Lo and Olszewski (2018) and Gordon et al. (2019) established some partial convergence results for a general class of dynamic learning processes, which includes the best-response dynamics. Establishing the convergence in the Crawford-Sobel model was indeed the main initial goal of this project. However, I will also provide other applications. In addition, my result directly covers only some settings, but the method of proof is potentially useful in other settings.

## 2 Preliminaries

The following proposition is extremely useful in elementary economic analysis.

**Proposition 1** *Let  $f : [0, 1] \rightarrow [0, 1]$  be an increasing (not necessarily continuous) function. For any  $x^0 \in [0, 1]$ , let  $x^k = f^k(x^0)$  for  $k = 1, 2, \dots$ , or in other words, let  $x^k$  be defined by induction as  $f(x^{k-1})$ . Then, the sequence  $(x^k)_{k=0}^\infty$  converges to some  $x^* \in [0, 1]$ .*

*In addition, if  $f$  is continuous at  $x^*$ , then  $x^*$  is a fixed point of  $f$ .*

Here is one out of many applications of this proposition.

**Example 1** *Consider a Cournot duopoly in which firms have convex cost functions, and face concave inverse demand, which decreases with their total output. Suppose that firms start by producing outputs  $q_1^0$  and  $q_2^0$ , and alternate in adjusting their outputs so that the current output of firm 1 in odd periods is the best response to the output of firm 2 from the previous period, and the current output of firm 2 in even periods is the best response to the output of firm 1 from the previous period.*

*By the concavity of inverse demand, the output  $q_1^k$  of firm 1 in any odd period  $2k + 1$  is an increasing function of its output  $q_2^{k-1}$  in period  $2k - 1$ . Similarly, the output  $q_2^k$  of firm 2 in any even period  $2k + 2$  is an increasing function of its output  $q_1^{k-1}$  in period  $2k$ . So, by Proposition 1 the outputs converge to the unique Cournot-Nash quantities.*

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<sup>3</sup>If  $x_i^{k-1} \geq x_i^k$  and  $x_i^k \leq x_i^{k+1}$  or  $x_i^{k-1} \leq x_i^k$  and  $x_i^k \geq x_i^{k+1}$ , and only one of the two inequalities is strict, the coordinate counts to the index as a half.

This is also well known that Proposition 1 is no longer true for decreasing functions. For example, if the function is defined by letting

$$f_1(x) = 1 - x,$$

or

$$f_2(x) = \begin{cases} 1 - x/2 & \text{for } x \leq 1/2; \\ 1/2 - x/2 & \text{for } x > 1/2, \end{cases}$$

then the sequence  $(x^k)_{k=0}^\infty$  does not converge for any  $x^0 \neq 1/2$ . In the former case, the function is continuous, and so has a fixed point. In the latter case, the function does not have a fixed point, and the sequence  $(x^k)_{k=0}^\infty$  does not converge even for  $x^0 = 1/2$ .

In this note, I provide a counterpart of Proposition 1 in more than one dimension, and some applications of this more general result in economics. In particular, I will be interested only in increasing functions, that is, functions  $f = (f_1, \dots, f_n) : S \subset [0, 1]^n \rightarrow S$  such that if  $x_1 \leq y_1, \dots, x_n \leq y_n$ , then  $f_1(x) \leq f_1(y), \dots, f_n(x) \leq f_n(y)$ , where  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ .

Notice that monotonicity (with respect to the coordinate-by-coordinate order) alone is insufficient for the convergence of sequences of iterations.

**Example 2** *Let  $f : [0, 1]^2 \rightarrow [0, 1]^2$  be the reflection of the square  $[0, 1]^2$  across the 45 degree line. This  $f$  is increasing, but the sequence  $(x^k)_{k=0}^\infty$  does not converge for any  $x^0$  out of the 45-degree line. In addition, one can easily modify this example, by using an idea similar to that from my previous example, to obtain an increasing function that has no fixed point, and such that the sequence  $(x^k)_{k=0}^\infty$  does not converge even for any  $x^0$  from the 45-degree line.*

The reason for the lack of convergence is that in one dimension,  $x^1 = f(x^0) \leq x^0$  implies that  $x^2 = f(x^1) \leq x^1$  (or  $x^1 = f(x^0) \geq x^0$  implies that  $x^2 = f(x^1) \geq x^1$ ), and by induction that the sequence  $(x^k)_{k=0}^\infty$  is monotone. And one of inequalities  $f(x^0) \leq x^0$  or  $f(x^0) \geq x^0$  must hold. In more than one dimension, there may be no “ $\leq$ ” or “ $\geq$ ” relation between  $f(x^0)$  and  $x^0$ , as it happened in Example 2.

One easy way of obtaining a counterpart of Proposition 1 in many dimensions is to assume monotonicity coordinate by coordinate. That is, to assume that for every  $i$ , if  $x_i \leq y_i$  then  $f_i(x) \leq f_i(y)$ . This implies that  $(x_i^k)_{k=0}^\infty$  is monotone, and so convergent, for all  $i$ . However, this result cannot have any interesting new applications. I will provide a more advanced result, which has interesting economic applications.

### 3 Result

To formulate my result, I must introduce two auxiliary concepts.

**Definition 1** *Let  $T^n = \{(x_1, \dots, x_n) \in [0, 1]^n : x_1 \leq \dots \leq x_n\}$ . A mapping  $f = (f_1, \dots, f_n) : S \subset T^n \rightarrow S$  is cutoff determined if there exist (“cutoff”) functions  $t_0, \dots, t_n : T^2 \rightarrow [0, 1]$  such that: (i)  $t_i(x_i, x_{i+1}) \in [x_i, x_{i+1}]$  for  $i = 0, \dots, n$ ; (ii)  $f_i(x_1, \dots, x_n)$  is determined by  $t_{i-1}(x_{i-1}, x_i)$  and  $t_i(x_i, x_{i+1})$ .*

In the definition above, and in what follows, I assume that  $x_0 \equiv 0$  and  $x_{n+1} \equiv 1$ .

According to this definition, the  $i$ -th coordinate of  $f$  is determined by  $x_i$  and its neighbor coordinates  $x_{i-1}$  and  $x_{i+1}$ ; in addition,  $x_{i-1}$  and  $x_i$ , and  $x_i$  and  $x_{i+1}$  first determine the ‘‘cutoffs’’  $t_{i-1}(x_{i-1}, x_i)$  and  $t_i(x_i, x_{i+1})$ , respectively, and then, the cutoffs  $t_{i-1}(x_{i-1}, x_i)$  and  $t_i(x_i, x_{i+1})$  determine  $f_i(x_1, \dots, x_n)$ .

**Definition 2** *A cutoff-determined mapping is neighbor-coordinate increasing if functions  $t_1, \dots, t_{n-1}$  are strictly increasing in each argument, and  $f_i$  strictly increases with  $t_{i-1}$  and strictly increases with  $t_i$ .*

**Remark 1** *There are three alternative formulations of Definitions 1 and 2, which respectively require:*

(1) *the existence of cutoff functions  $t_1, \dots, t_{n-1} : T^2 \rightarrow [0, 1]$  with the same properties, and in which I assume that  $t_0 \equiv 0$  and  $t_n \equiv 1$ ;*

(2) *the existence of cutoff functions  $t_0, \dots, t_{n-1} : T^2 \rightarrow [0, 1]$  with the same properties, and in which I assume that  $x_0 \equiv 0$  and  $t_n \equiv 1$ ;*

(3) *the existence of cutoff functions  $t_1, \dots, t_n : T^2 \rightarrow [0, 1]$  with the same properties, and in which I assume that  $t_0 \equiv 0$  and  $x_{n+1} \equiv 1$ .*

*Theorem 1 below holds in each of the three alternative cases.*

**Theorem 1** *Suppose that a cutoff-determined mapping  $f = (f_1, \dots, f_n) : S \subset T^n \rightarrow S$  is neighbor-coordinate increasing. For any  $x^0 \in S$ , let  $x^k = f^k(x^0)$  for  $k = 1, 2, \dots$ , or in other words, let  $x^k$  be defined by induction as  $f(x^{k-1})$ .*

*Then, the sequence  $(x^k)_{k=0}^\infty$  converges to some  $x^* \in T^n$ . In addition, if  $x^* \in S$ , and function  $f$  is continuous at  $x^*$ , then  $x^*$  is a fixed point of  $f$ .*

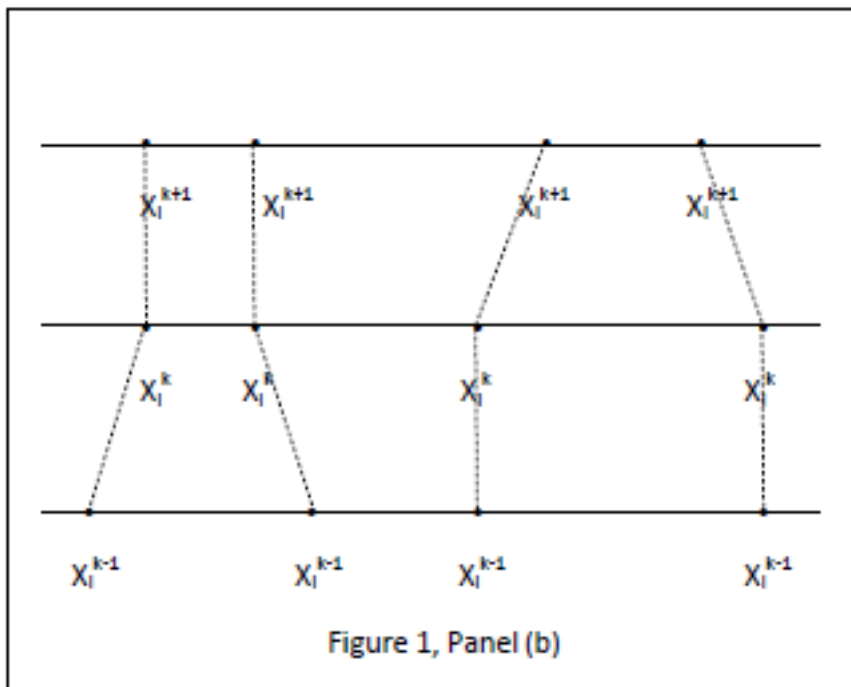
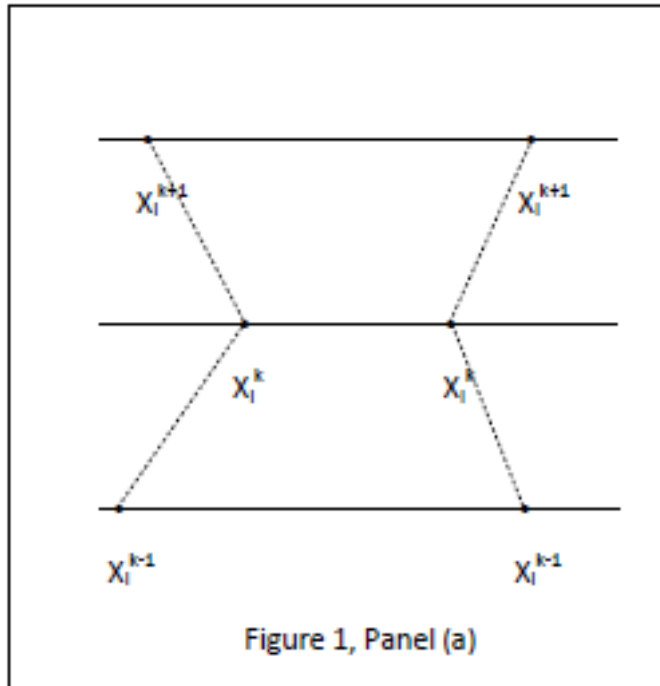
When  $n = 1$  and  $S = T^1$ , Theorem 2 reduces to Theorem 1. I postpone the proof to Appendix. However, I will explain the key idea now. This idea is contained in Lemma 1 (below), which relies on the following concepts.

**Definition 3** *A coordinate  $l = 1, \dots, n$  is called reversal in period  $k$  if (i)  $x_l^{k-1} < x_l^k > x_l^{k+1}$ , or (ii)  $x_l^{k-1} > x_l^k < x_l^{k+1}$ . If one of the inequalities in (i) or in (ii) is replaced with equality, and the other inequality stays strict, then the coordinate is called half-reversal.*

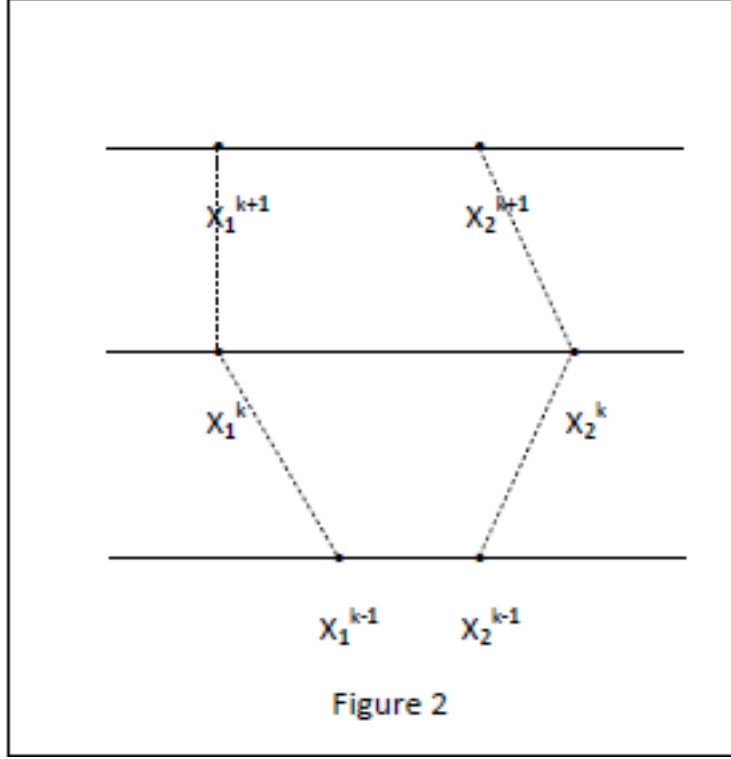
This definition is illustrated in Figure 1. The next two definitions are illustrated in Figure 2.

**Definition 4** *A pair of neighbor coordinates  $l$  and  $l+1$  is called incompatible in period  $k$  if (i)  $x_l^k < x_l^{k+1}$  and  $x_{l+1}^{k+1} < x_{l+1}^k$ ; or (ii)  $x_l^{k+1} < x_l^k$  and  $x_{l+1}^k < x_{l+1}^{k+1}$ . If one of the inequalities in (i) or in (ii) is replaced with equality, and the other inequality stays strict, then the pair is called half-incompatible. Otherwise, coordinates  $l$  and  $l+1$  are called compatible.*

**Definition 5** *The incompatibility in period  $k$  is equal to the number of incompatible pairs (among 1 and 2, 2 and 3, ...,  $n-1$  and  $n$ ) plus a half of the number of half-incompatible pairs.*



**Figure 1.** Panel (a) depicts two possible types of a reversal coordinate  $l$ , and panel (b) depicts four possible types of a half-reversal coordinate.



**Figure 2.** Pair 1 and 2 is incompatible in period  $k - 1$ , and is half-incompatible in period  $k$ .

So, if  $n = 2$ , the incompatibility is equal to 1 in period  $k - 1$ , and is equal to  $1/2$  in period  $k$ .

**Lemma 1** (a) *If coordinate  $n$  is reversal, or is half-reversal, then the incompatibility in period  $k - 1$  is strictly higher than that in period  $k$ .*

(b) *If coordinate  $n$  is neither reversal nor half-reversal, then the incompatibility at period  $k - 1$  is no lower than that in period  $k$ .*

To give some flavor of the proof, I will first show this lemma for  $n = 2$ . Suppose first that  $x_2^{k-1} < x_2^k > x_2^{k+1}$  (a case in part (a)). Since  $x_3 \equiv 1$ ,  $t_2(x_2^k, x_3^k) > t_2(x_2^{k-1}, x_3^{k-1})$  by neighbor-coordinate monotonicity. Also by neighbor-coordinate monotonicity, this inequality together with  $x_2^k > x_2^{k+1}$  imply that  $t_1(x_1^k, x_2^k) < t_1(x_1^{k-1}, x_2^{k-1})$ . In turn, this together with  $x_2^{k-1} < x_2^k$  imply that  $x_1^k < x_1^{k-1}$  (again by neighbor-coordinate monotonicity). So, pair 1 and 2 must be incompatible in period  $k - 1$ . In addition, since  $x_0 \equiv 0$ , it must be that  $t_0(x_0^k, x_1^k) < t_0(x_0^{k-1}, x_1^{k-1})$  and  $t_0 \equiv 0$  imply that  $x_1^{k+1} < x_1^k$ . And  $t_0(x_0^k, x_1^k) < t_0(x_0^{k-1}, x_1^{k-1})$  together with  $t_1(x_1^k, x_2^k) < t_1(x_1^{k-1}, x_2^{k-1})$  imply that  $x_1^{k+1} < x_1^k$ . So, since  $x_2^k > x_2^{k+1}$ , pair 1 and 2 is compatible in period  $k$ . Thus, the incompatibility in period  $k - 1$  is equal to 1, and the incompatibility in period  $k$  is equal to 0.

Suppose now that  $x_2^{k-1} < x_2^k < x_2^{k+1}$  (a case in part (b)). Since  $x_3 \equiv 1$ ,  $t_2(x_2^k, x_3^k) > t_2(x_2^{k-1}, x_3^{k-1})$  by neighbor-coordinate monotonicity. Also by neighbor-coordinate monotonicity, this together with  $x_2^k < x_2^{k+1}$

imply that  $t_1(x_1^k, x_2^k) < t_1(x_1^{k-1}, x_2^{k-1})$ . This together with  $x_2^{k-1} < x_2^k$  imply that  $x_1^k < x_1^{k-1}$ . So, since  $x_2^{k-1} < x_2^k$ , pair 1 and 2 is incompatible in period  $k - 1$ . It is not necessary for the argument, but useful to notice that since  $x_0 \equiv 0$  and  $x_1^k < x_1^{k-1}$ ,  $t_0(x_0^k, x_1^k) < t_0(x_0^{k-1}, x_1^{k-1})$ . Together with  $t_1(x_1^k, x_2^k) < t_1(x_1^{k-1}, x_2^{k-1})$ , this implies that  $x_1^{k+1} < x_1^k$ . So, pair 1 and 2 is also incompatible in period  $k$ . Thus, the incompatibility both in period  $k - 1$  and in period  $k$  is equal to 1.

All other cases (for  $n = 2$ ) rely on analogous arguments. Lemma 1 implies that  $(x_n^k)_{k=K}^\infty$  is monotonic for large enough values of  $K$ , and so convergent. By repeating the arguments, I prove by induction with respect to  $l = n, \dots, 1$  that the sequence  $(x_l^k)_{k=K}^\infty$  is monotonic for large enough values of  $K$ , and so convergent for all  $l = 1, \dots, n$ .

The proof becomes more involved for  $n > 2$ , so the proof of Lemma 1 for  $n = 3$  provides better intuition. Consider first the case when  $x_3^{k-1} < x_3^k > x_3^{k+1}$ . Since  $x_4 \equiv 1$ ,  $t_3(x_3^{k-1}, x_4^{k-1}) < t_3(x_3^k, x_4^k)$ . This, as all subsequent inequalities, follow from neighbor-coordinate monotonicity. Thus,  $x_3^k > x_3^{k+1}$  implies that  $t_2(x_2^k, x_3^k) < t_2(x_2^{k-1}, x_3^{k-1})$ . And together with  $x_3^{k-1} < x_3^k$ , this implies that  $x_2^k < x_2^{k-1}$ . So, pair 2 and 3 is incompatible in period  $k - 1$ .

If  $x_2^{k+1} < x_2^k$ , then pair 2 and 3 is compatible in period  $k$ . Thus, the incompatibility in period  $k$  is lower than that in period  $k - 1$ , unless pair 1 and 2 is incompatible in period  $k$ , but is compatible in period  $k - 1$ . However, this means that  $x_1^k < x_1^{k+1}$  and  $x_1^k < x_1^{k-1}$ . The latter inequality, together with  $x_2^k < x_2^{k-1}$ , imply that  $t_1(x_1^k, x_2^k) < t_1(x_1^{k-1}, x_2^{k-1})$ . Thus, the former inequality can hold true only when  $t_0(x_0^{k-1}, x_1^{k-1}) < t_0(x_0^k, x_1^k)$ . However, the latter inequality, together with  $x_0 \equiv 0$ , imply to the contrary that  $t_0(x_0^{k-1}, x_1^{k-1}) > t_0(x_0^k, x_1^k)$ .

If  $x_2^{k+1} \geq x_2^k$ , then  $t_1(x_1^k, x_2^k) > t_1(x_1^{k-1}, x_2^{k-1})$ , because  $t_2(x_2^k, x_3^k) < t_2(x_2^{k-1}, x_3^{k-1})$ . This in turn implies that  $x_1^{k-1} < x_1^k$ , because  $x_2^k < x_2^{k-1}$ . Thus,  $t_0(x_0^{k-1}, x_1^{k-1}) < t_0(x_0^k, x_1^k)$ , because  $x_0 \equiv 0$ . This in turn implies that  $x_1^k < x_1^{k+1}$ , because  $t_1(x_1^{k-1}, x_2^{k-1}) < t_1(x_1^k, x_2^k)$ . Therefore, the incompatibility in period  $k - 1$  is equal to 2, and the incompatibility in period  $k$  is equal to 1. (The latter claim holds true, because pair 1 and 2 is compatible and pair 2 and 3 is incompatible if  $x_2^{k+1} > x_2^k$ , and both pairs are half-incompatible if  $x_2^{k+1} = x_2^k$ .)

Consider now the case when  $x_3^{k-1} < x_3^k < x_3^{k+1}$ . Since  $x_4 \equiv 1$ ,  $t_3(x_3^{k-1}, x_4^{k-1}) < t_3(x_3^k, x_4^k)$ . Suppose first that  $x_2^{k-1} < x_2^k$ . Then  $t_2(x_2^{k-1}, x_3^{k-1}) < t_2(x_2^k, x_3^k)$ , because  $x_3^{k-1} < x_3^k$ . If in addition  $x_1^{k-1} < x_1^k$ , then  $t_1(x_1^{k-1}, x_2^{k-1}) < t_1(x_1^k, x_2^k)$ , because  $x_2^{k-1} < x_2^k$ , and  $t_0(x_0^{k-1}, x_1^{k-1}) < t_0(x_0^k, x_1^k)$ , because  $x_0 \equiv 0$ . Thus,  $x_2^k < x_2^{k+1}$  and  $x_1^k < x_1^{k+1}$ , and so the incompatibility both in period  $k$  and in period  $k - 1$  is 0. If in addition  $x_1^{k-1} = x_1^k$ , then  $t_1(x_1^{k-1}, x_2^{k-1}) < t_1(x_1^k, x_2^k)$ , because  $x_2^{k-1} < x_2^k$ , and  $t_0(x_0^{k-1}, x_1^{k-1}) = t_0(x_0^k, x_1^k)$ , because  $x_0 \equiv 0$ . Thus,  $x_2^k < x_2^{k+1}$  and  $x_1^k < x_1^{k+1}$ , and so the incompatibility in period  $k$  is 0 and that in period  $k - 1$  is 1/2. Finally, if in addition  $x_1^{k-1} > x_1^k$ , then the incompatibility in period  $k - 1$  is 1. The incompatibility in period  $k$  can be higher than 1 only when  $x_2^k > x_2^{k+1}$ . (Otherwise, 2 and 3 is compatible if  $x_2^k < x_2^{k+1}$ , or both 1 and 2, and 2 and 3 are half-incompatible if  $x_2^k = x_2^{k+1}$ .) However, this implies that  $t_1(x_1^{k-1}, x_2^{k-1}) > t_1(x_1^k, x_2^k)$ , because  $t_2(x_2^{k-1}, x_3^{k-1}) < t_2(x_2^k, x_3^k)$ . Since  $t_0(x_0^{k-1}, x_1^{k-1}) > t_0(x_0^k, x_1^k)$  by  $x_1^{k-1} > x_1^k$  and  $x_0 \equiv 0$ , we have that  $x_1^k > x_1^{k+1}$ . This implies that pair 1 and 2 is compatible in period  $k$ ,

and so the incompatibility in period  $k$  is 1.

Suppose now that  $x_2^{k-1} = x_2^k$ . Then  $t_2(x_2^{k-1}, x_3^{k-1}) < t_2(x_2^k, x_3^k)$ , because  $x_3^{k-1} < x_3^k$ . If in addition  $x_1^{k-1} < x_1^k$ , then the incompatibility in period  $k-1$  is 1. (Both 1 and 2, and 2 and 3 are half-incompatible.) Further,  $t_1(x_1^{k-1}, x_2^{k-1}) < t_1(x_1^k, x_2^k)$ , and so  $x_2^k < x_2^{k+1}$ , because  $t_2(x_2^{k-1}, x_3^{k-1}) < t_2(x_2^k, x_3^k)$ . Thus, the incompatibility in period  $k$  is at most 1, because pair 2 and 3 is compatible. If in addition  $x_1^{k-1} = x_1^k$ , then  $t_1(x_1^{k-1}, x_2^{k-1}) = t_1(x_1^k, x_2^k)$  and  $t_0(x_0^{k-1}, x_1^{k-1}) = t_0(x_0^k, x_1^k)$ . This implies that  $x_2^{k+1} > x_2^k$ , because  $t_3(x_3^{k-1}, x_4^{k-1}) < t_3(x_3^k, x_4^k)$ , and that  $x_1^{k+1} = x_1^k$ . Thus, the incompatibility both in period  $k-1$  and in period  $k$  is  $1/2$ . If in addition  $x_1^{k-1} > x_1^k$ , then  $t_1(x_1^{k-1}, x_2^{k-1}) > t_1(x_1^k, x_2^k)$ . Further,  $t_0(x_0^{k-1}, x_1^{k-1}) > t_0(x_0^k, x_1^k)$ , because  $x_0 \equiv 0$ . So,  $x_1^{k+1} < x_1^k$ . The incompatibility in period  $k-1$  is 1. (Both 1 and 2, and 2 and 3 are half-incompatible.) So is the incompatibility in period  $k$ . (If  $x_2^{k+1} > x_2^k$ , then 1 and 2 is incompatible, but 2 and 3 is compatible. If  $x_2^{k+1} < x_2^k$ , then 1 and 2 is compatible, but 2 and 3 is incompatible. If  $x_2^{k+1} = x_2^k$ , then both pairs are half-incompatible.)

Suppose finally that  $x_2^{k-1} > x_2^k$ . Then, the incompatibility in period  $k$  cannot be higher than that in period  $k-1$ , unless  $x_2^k > x_2^{k+1}$ . Otherwise, since  $x_3^k < x_3^{k+1}$ , the incompatibility in period  $k$  cannot be higher than 1, and the incompatibility in period  $k-1$  is at least 1. Further,  $x_1^{k-1} \geq x_1^k$ , because otherwise the incompatibility in period  $k-1$  would be equal to 2, and the incompatibility in period  $k$  cannot be higher than that. Thus,  $t_1(x_1^{k-1}, x_2^{k-1}) > t_1(x_1^k, x_2^k)$ , and since  $x_0 \equiv 0$ ,  $t_0(x_0^{k-1}, x_1^{k-1}) \geq t_0(x_0^k, x_1^k)$ . The last two inequalities imply that  $x_1^k > x_1^{k+1}$ . Since also  $x_2^k > x_2^{k+1}$ , the incompatibility in period  $k$  is equal to 1, and this completes the proof.

## 4 Applications

### 4.1 The Crawford-Sobel model

Consider the celebrated Crawford-Sobel (1982) model of information transmission. In their model, there are two players: the sender and the receiver. The sender obtains a signal, sends a message to the receiver, and the receiver takes an action. Let  $T = [0, 1]$  represent the set of possible signals of the sender, which I will also call types, let  $A = [0, 1]$  represents the set of possible actions of the receiver, and let  $M$  be the set of allowed messages. A signal is drawn according to the prior probability distribution with a positive density  $\varphi$ . Finally, let  $U_S(t, a)$  and  $U_R(t, a)$  be the payoff functions of the two players. The payoffs depend on the signal and the action, but not on the message. Assume, as Crawford and Sobel do, that  $U_S(t, a)$  and  $U_R(t, a)$  are twice continuously differentiable, and that  $\partial^2 U_i(t, a)/\partial a^2 < 0$  and  $\partial^2 U_i(t, a)/\partial a \partial t > 0$ , where  $i = S, R$ .

By the assumption that  $\partial^2 U_i(t, a)/\partial a^2 < 0$ ,

$$y^i(t) = \arg \max_{a \in A} U_i(t, a),$$

is uniquely defined for  $i = S, R$ . As Crawford and Sobel, I assume that

$$b := \min_{t \in T} (y^S(t) - y^R(t)) > 0.$$



Let

$$y^R(t_l, t_h) = \arg \max_{a \in A} \int_{t_l}^{t_h} U_R(t, a) f(t) dt$$

be the receiver's best response if she believes that the sender's types is distributed according to the prior truncated to interval  $[t_l, t_h]$ , and for each pair of actions  $a_l < a_h$ , let  $x^S(a_l, a_h)$  be the sender's type which is indifferent between actions  $a_l$  and  $a_h$ . By the assumption that  $\partial^2 U_S(t, a) / \partial a \partial t > 0$ , all types to the left of  $x^S(a_l, a_h)$  prefer  $a_l$  and all types to the right of  $x^S(a_l, a_h)$  prefer  $a_h$ . It may happen that type 0 strictly prefers  $a_h$  to  $a_l$ , that is, such an indifferent type does not exist. But since  $b > 0$ , type 1 strictly prefers  $a_h$  to  $a_l$  for any  $a_l < a_h$ , so such an indifferent type always exists if type 0 weakly prefers  $a_l$  to  $a_h$ . Because  $\partial^2 U_R(t, a) / \partial a \partial t > 0$ ,  $y^R(t_l, t_h)$  increases in both arguments. So does  $x^S(a_l, a_h)$ .

Under the best-response dynamics, I constrain the sender to using a finite number of messages  $m_1, m_2, \dots, m_N$ .<sup>4</sup> The receiver takes in period 0 actions  $a_1^0 \leq a_2^0 \leq \dots \leq a_N^0$  in response to these messages. The sender, contingent on her signal, sends a message  $m_i$  such that  $a_i^0$  maximizes her payoff. Some types of the sender have multiple optimal messages. So, I must specify the way in which the sender breaks ties.

Only one type of the sender can be indifferent between two messages inducing different actions. The way in which this type breaks the tie is inessential. However, for some intervals of the sender's types, there may exist multiple messages inducing their optimal response. Then, any subset of such an interval can choose a specific message which induces this response. This clearly requires imposing some regularity condition on tie-breaking. Without any regularity condition, there would be no chance to establish any convergence result.

I assume that tie-breaking is *monotonic*. This assumption can be formulated as follows:

Suppose that more than one message induces an action  $a$ . Denote by  $x^S(a)$  the type of the sender for whom  $a$  is the most preferred action (across all actions from  $[0, 1]$ ). Monotonicity requires the types lower than  $x^S(a)$  who most prefer action  $a$  among all induced responses of the receiver send the same message (one of those which induce  $a$ ), and the types higher than  $x^S(a)$  who most prefer action  $a$  also send the same message (again, one of those which induce  $a$ ). In addition, the two messages must be different. Type  $x^S(a)$  randomizes with full support over the other messages inducing action  $a$ . Of course,  $x^S(a)$  does not exist if the bliss point of type 0's preferences is higher than  $a$ . In this case, monotonicity requires that all types except 0 who most prefer action  $a$  among all induced responses of the receiver send the same message (one of those which induce  $a$ ). Type 0 randomizes with full support over the other messages inducing action  $a$ .

Notice that for any given responses of the receiver, the partition of the sender's types generated by those responses under the monotonic tie-breaking is the limit of the partitions generated by pairwise distinct responses when they converge to the given responses.

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<sup>4</sup>This assumption is without loss of generality, because any equilibrium outcome can be replicated in an equilibrium in which only a finite (bounded across all equilibria) number of messages is used by the sender.

Return now to the definition the best-response dynamics. In period 1, the receiver best responds to each message given the sender's strategy from period 0, and the sender best responds to the receiver's strategy from period 1. Notice that by monotonicity each message is sent by some of the sender's types, except possibly messages  $m_1, m_2, \dots, m_l$  for some  $l < N$ , which are sent by no type. In addition, the set of the sender's types sending the message is a (possibly degenerated) interval. In each subsequent period, the receiver best responds to each message given the sender's strategy from the previous period, and the sender monotonically best responds to the receiver's strategy from the current period. Finally, I assume that the receiver responds to messages  $m_1, m_2, \dots, m_l$  as if they were sent only by type zero.

Crawford and Sobel (1982) characterized the equilibria of their model. Typically, the model has multiple equilibria. They showed that in each equilibrium the interval of types is partitioned into a finite number of subintervals such that all types from the interior of each subinterval induce the same action of the receiver. In addition, the number of partition subintervals must be bounded across all equilibria by a function of parameter  $b$ .

Theorem 2 implies that:

**Corollary 1** *For any set of initial actions  $a_1^0 \leq a_2^0 \leq \dots \leq a_N^0$ , the strategies of the sender and the receiver converge to an equilibrium of the Crawford-Sobel model.*

Note that the corollary does not say anything about to which equilibrium the strategies will converge to. They can converge to any equilibrium. Indeed, if the initial set of actions coincides with the actions in some equilibrium, then the strategies of the players are constant over time, and so converge to this equilibrium. More generally, one could try to characterize the basins of attraction of all equilibria. This is, however, a very demanding task, and I can provide no answer to this question in the general case. I will characterize the basins of attraction in Section 5, but only in the special "uniform-quadratic" case, in which  $U_S(t, a) = -(a - t - b)^2$ ,  $U_R(t, a) = -(a - t)^2$  and  $\varphi$  is uniform.

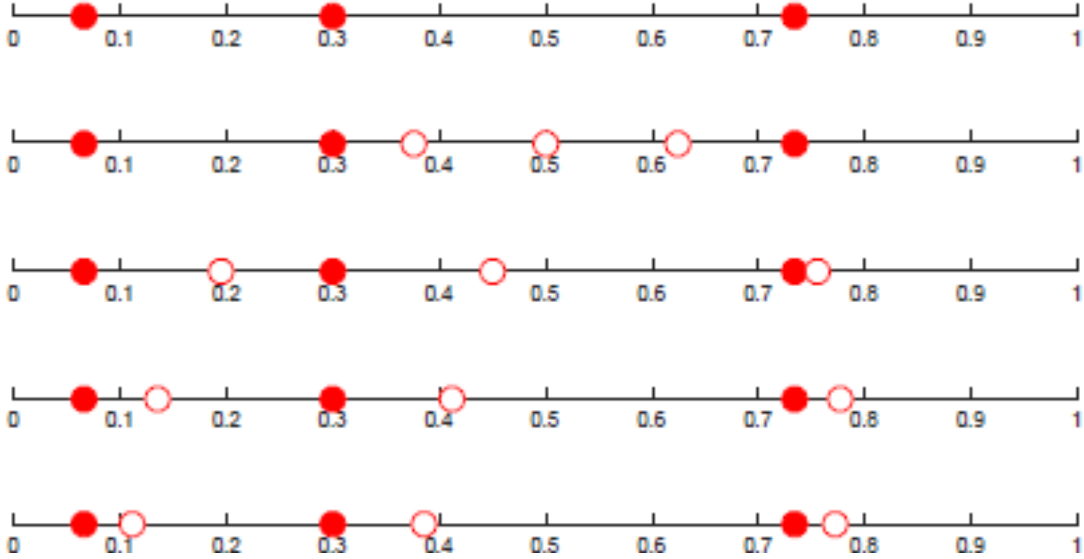
**Proof.** Apply Theorem 1 to  $f : S \rightarrow S$ , where  $S = \{(a_1, \dots, a_N) \in [0, 1]^N : a_1 \leq \dots \leq a_N\}$ , and  $f(a_1, \dots, a_N)$  are the receiver's best responses to the sender's strategy that itself best responds to the receiver taking actions  $a_1, \dots, a_N$ .

Recall that the sender's best responses are assumed to be monotonic. This assumption guarantees the existence of the cutoff functions, because the set of types sending any given message is a (possibly degenerated) interval. So, mapping  $f$  is cutoff-determined.

And since  $x^S(a_l, a_h)$  and  $y^R(t_l, t_h)$  are increasing (with respect to both arguments), mapping  $f$  is also neighbor-coordinate increasing.

■

As an example, consider  $U_S(t, a) = -(a - t - 0.05)^2$ ,  $U_R(t, a) = -(a - t)^2$ ,  $f \equiv 1$ ,  $a_1^0 = 0.375$ ,  $a_2^0 = 0.5$ ,  $a_3^0 = 0.625$ . Then,  $a_3$  increases in the first two periods, while  $a_1$  and  $a_2$  decrease, and beginning from period 3, all three actions decrease, and gradually converge to the equilibrium with actions  $a_1 = 0.066667$ ,  $a_2 = 0.3$ ,  $a_3 = 0.733333$ . The first three iterations are depicted in Figure 3.



**Figure 3.** The equilibrium actions in solid red, and the 3 actions in periods 0, 1, 2, 3 (going down) in empty red.

As I have said in Introduction, establishing the convergence of best-response dynamics in the Crawford-Sobel model is the main application of the paper. However, my result applies to a variety of similar settings. The next two subsections contain other applications.

## 4.2 A version of Hotelling (1929) linear city model

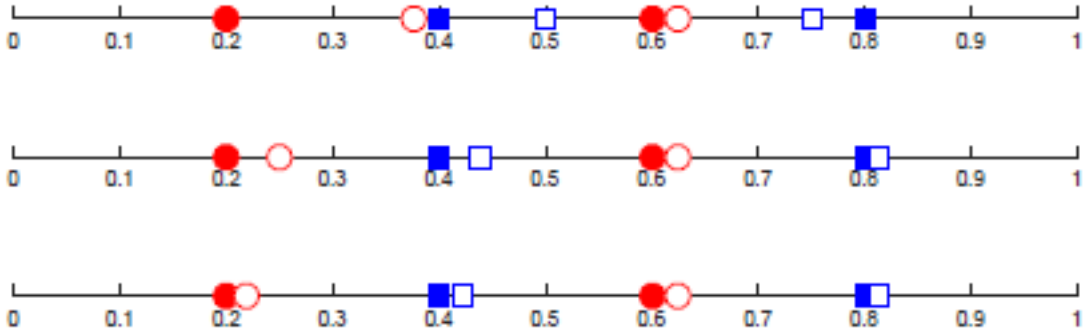
Suppose that two firms locate  $n$  outlets each on the interval  $[0, 1]$ . The outlets of firm 1 are initially (in period 0) located at points  $x_1, x_3, \dots, x_{2n-1}$ , and the outlets of firm 2 are initially located at points  $x_2, x_4, \dots, x_{2n}$ , where  $0 < x_1 < x_2 < \dots < x_{2n-1} < x_{2n} < 1$ .

In odd periods, firm one can move its outlets, and in even periods, firm 2 can move its outlets. When firms decide where to move their outlets, they best respond to the current locations of the outlets of their opponents. Suppose that each firm faces some cost of moving away from an outlet of the opponent, or away from the endpoint of the interval  $[0, 1]$ . More specifically, suppose that the cost of locating an outlet at point

$x$ , between the outlets  $x_l < x$  and  $x_h > x$  of the opponent (or between  $x_l = 0$  and  $x_h > x$ , or between  $x_l < x$  and  $x_h = 1$ ) is the sum of two costs: one depends on the distance between  $x_l$  and  $x$ , and the other depends on the distance between  $x_h$  and  $x$ . The marginal costs are zero at  $x_l$  and  $x_h$ , respectively, and increase with the distance. There is a variety of reasons for which the costs may have this structure. For example, if a firm moves its location away from a location of the opponent, some customers who “live” between the two locations will switch to buying from the opponent, and others will stop buying at all. Suppose also that it makes no sense to locate more than one outlet between a pair of neighbor outlets of the opponent. For example, the firm would then “cannibalize” some of its own customers. Given these assumptions, it is optimal for each firm to locate outlets at the points in which the marginal costs (that of moving away from  $x_l$ , and that of moving away from  $x_h$ ) are equal.

A sample evolution of locations over time of two outlets of each firm, for the marginal costs being equal to the distance between  $x_l$  and  $x$ , and the distance between  $x_h$  and  $x$  (or the distance  $x$  and an endpoint) looks as follows: Suppose that:  $x_1^0 = 3/8$ ,  $x_2^0 = 1/2$ ,  $x_3^0 = 5/8$ ,  $x_4^0 = 3/4$ . First, firm 1 moves its lower outlet down to  $x_1^1 = 1/4$ , not moving its higher outlet, which stays at  $x_3^1 = 5/8$ . Then, firm 2 moves its lower outlet down to  $x_2^1 = 7/16$ , and its upper outlet up to  $x_4^1 = 13/16$ .

In the second round, firm 1 moves its lower outlet further down to  $x_1^2 = 7/32$ , not moving its higher outlet, which stays at  $x_3^2 = 5/8$ . Then, firm 2 moves its lower outlet further down to  $x_2^2 = 27/64$ , not moving its upper outlet, which stays at  $x_4^2 = 13/16$ . From the next period on, both firms move their both outlets down in each period, and gradually converge to the equilibrium locations:  $x_1^\infty = 1/5$ ,  $x_2^\infty = 2/5$ ,  $x_3^\infty = 3/5$ ,  $x_4^\infty = 4/5$ .



**Figure 4.** The equilibrium locations: of firm 1 represented by solid red discs, of firm 2 by solid blue squares, and the locations in periods 0, 1, 2 represented by empty red discs and by empty blue squares, respectively.

Theorem 2 implies that the locations must converge over time to a stable configuration, in which firms best respond to their opponents by staying in their locations. To see why, consider the mapping  $f =$

$(f_1, f_3, \dots, f_{2n-1})$  from the set

$$T^n = \{(x_1, x_3, \dots, x_{2n-1}) : 0 \leq x_1 \leq x_3 \leq \dots \leq x_{2n-1} \leq 1\}$$

into itself, defined as follows: Given a vector  $(x_1, x_3, \dots, x_{2n-1})$  such that  $0 \leq x_1 \leq x_3 \leq \dots \leq x_{2n-1} \leq 1$ , let  $x_{2k} = t_{2k}(x_{2k-1}, x_{2k+1})$ , for  $k = 1, \dots, n$ , be the optimal location of the  $k$ -th outlet of firm 2. Then, let  $f_{2k-1}(x_1, x_3, \dots, x_{2n-1})$ , for  $k = 1, \dots, n$ , be the optimal location of the  $k$ -th outlet of firm 1, given the vector  $(x_2, x_4, \dots, x_{2n})$ . The convergence to a stable configuration follows now from observation (3) in Remark 1.

### 4.3 A version of Tiebout (1956) model of migration

There is a continuum of citizens are located on an interval, and their locations represent the bliss points of their preferences regarding the provision of a public good. The distribution of bliss points has a positive density, and is commonly known.

Suppose that the citizens are divided into  $n$  segments (districts), and that the provision of the public good in each district is determined by the Condorcet winner (selected by the citizens of the district). Suppose that the initial division (in period 0) is exogenous. Then, in each subsequent period  $t \geq 1$ , citizens may leave their districts and join other districts, whose Condorcet winners in period  $t - 1$  are closer to the bliss point of their preferences. Such shifts of citizens between districts affect the Condorcet winners, that is, a new set of Condorcet winners is selected. Then, citizens may again leave their districts and join other districts. This process is continued ad infinitum.

Theorem 2 implies that the partition must converge over time to a stable partition, in which no citizen has an incentive to leave her/his district for joining another one. The proof is analogous, but simpler to the proof of Corollary 1, and uses the mapping  $f = (f_1, \dots, f_n)$  from the set

$$S = \{(x_1, \dots, x_n) : 0 \leq x_1 < x_2 < \dots < x_n \leq 1\}$$

into itself, defined as follows. For every  $x_1, \dots, x_n$  which represent the current provisions of the districts, and  $i = 1, \dots, n - 1$ , let  $t_i(x_i, x_{i+1})$  be the bliss point of the citizen who is indifferent between the provisions  $x_i$  and  $x_{i+1}$ . Then,  $t_0, \dots, t_n$  (where  $t_0 \equiv 0$  and  $t_n \equiv 1$ ) determine the partition of the interval of all citizens into the segments consisting of the members of particular districts. Define  $f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)$  as the Condorcet winners selected by the members of each district.

The mapping  $f = (f_1, \dots, f_n)$  is obviously cutoff-determined and neighbor-coordinate increasing. Thus, the segments consisting of the members of particular districts converge over time to a stable partition. Note that I used here the observation (1) of Theorem 2 from Remark 1.

## 5 Basins of Attraction

In this section, I characterize basins of attraction, but only in some special case, namely, in the version of the Crawford-Sobel model in which the sender's type is distributed uniformly on interval  $[0, 1]$ , and the players'

utilities are:  $U_S(a, t) = -(a - t - b)^2$  and  $U_R(a, t) = -(a - t)^2$  for some  $b > 0$ . In this case, there is an  $N^* \geq 1$  such that for every  $N \leq N^*$  there exists a unique equilibrium with  $N$  partition intervals (i.e., with  $N$  equilibrium actions).

**Theorem 2** *If  $N < N^*$  is the number of messages, then no matter what actions these messages induce initially, then the play converges under the best-response dynamics to the equilibrium with  $N$  partition intervals. If  $N \geq N^*$ , then the play converges to the equilibrium with  $N^*$  partition intervals.*

**Proof.** The result is straightforward for  $N = 1$ , I will assume throughout the proof that  $N > 1$ . Assume first that  $N \leq N^*$ . I will describe at the end of the proof the changes that are required when  $N > N^*$ . Let  $c_1 \leq \dots \leq c_N$  be the equilibrium actions in the  $N$ -action equilibrium. For any set  $a_1 \leq \dots \leq a_N$  of  $N$  actions, let  $t(a_i, a_{i+1})$  be the type that is indifferent between the two neighbor actions  $a_i$  and  $a_{i+1}$  if such a type exists, and let it be type 0 otherwise; more precisely, let

$$t(a_i, a_{i+1}) = \begin{cases} \frac{a_i + a_{i+1}}{2} - b & \text{if } \frac{a_i + a_{i+1}}{2} - b \geq 0 \\ 0 & \text{if } \frac{a_i + a_{i+1}}{2} - b < 0 \end{cases}.$$

Given actions  $a_1 \leq \dots \leq a_N$  induced in period  $k$ , denote by  $a'_1 \leq \dots \leq a'_N$  the actions induced in period  $k + 1$ . Then:

$$a'_i = 0 \text{ if } i < N \text{ and } \frac{a_i + a_{i+1}}{2} - b < 0; \quad (1)$$

$$a'_i = \frac{a_i + a_{i+1}}{4} - \frac{b}{2} \text{ if } i = 1 \text{ and } \frac{a_i + a_{i+1}}{2} - b \geq 0; \quad (2)$$

$$a'_i = \frac{a_i + a_{i+1}}{4} - \frac{b}{2} \text{ if } 1 < i < N \text{ and } \frac{a_{i-1} + a_i}{2} - b < 0; \quad (3)$$

$$a'_i = \frac{a_{i-1} + 2a_i + a_{i+1}}{4} - b \text{ if } 1 < i < N \text{ and } \frac{a_{i-1} + a_i}{2} - b \geq 0; \quad (4)$$

$$a'_i = \frac{1}{2} + \frac{a_{i-1} + a_i}{4} - \frac{b}{2} \text{ if } i = N \text{ and } \frac{a_{i-1} + a_i}{2} - b \geq 0; \quad (5)$$

$$a'_i = \frac{1}{2} \text{ if } i = N \text{ and } \frac{a_{i-1} + a_i}{2} - b < 0. \quad (6)$$

Of course,  $c'_i = c_i$  for  $i = 1, \dots, N$ .

We will equip the set of vectors  $(a_1, \dots, a_N)$  such that  $a_1 \leq \dots \leq a_N$  with  $L^1$ -norm  $|a_1| + \dots + |a_N|$ , and show that

$$|c'_1 - a'_1| + \dots + |c'_n - a'_n| \leq \frac{|c_1 - a_1|}{2} + |c_2 - a_2| + \dots + |c_{N-1} - a_{N-1}| + \frac{|c_N - a_N|}{2}. \quad (7)$$

To show this inequality, I will estimate  $|c'_i - a'_i|$  for each  $i$ . The following six cases are exhaustive, because only cases (2), (4) and (5) can apply to  $c'_i$ :

Case 1. (1) applies to  $a'_i$  and (2) applies to  $c'_i$

$$|c'_i - a'_i| = \left| \frac{c_i + c_{i+1}}{4} - \frac{b}{2} \right| \leq \left| \frac{c_i + c_{i+1}}{4} - \frac{a_i + a_{i+1}}{4} \right| \leq \frac{|c_i - a_i|}{4} + \frac{|c_{i+1} - a_{i+1}|}{4}$$

Case 2. (2) applies to  $a'_i$  and (2) applies to  $c'_i$

$$|c'_i - a'_i| = \left| \frac{c_i + c_{i+1}}{4} - \frac{a_i + a_{i+1}}{4} \right| \leq \frac{|c_i - a_i|}{4} + \frac{|c_{i+1} - a_{i+1}|}{4}$$

Case 3. (1) applies to  $a'_i$  and (4) applies to  $c'_i$

$$\begin{aligned} |c'_i - a'_i| &= \left| \frac{c_{i-1} + 2c_i + c_{i+1}}{4} - b \right| \leq \left| \frac{c_{i-1} + c_i}{4} - \frac{b}{2} \right| + \left| \frac{c_i + c_{i+1}}{4} - \frac{b}{2} \right| \\ &\leq \left| \frac{c_{i-1} + c_i}{4} - \frac{a_{i-1} + a_i}{4} \right| + \left| \frac{c_i + c_{i+1}}{4} - \frac{a_i + a_{i+1}}{4} \right| \\ &\leq \frac{|c_{i-1} - a_{i-1}|}{4} + \frac{|c_i - a_i|}{2} + \frac{|c_{i+1} - a_{i+1}|}{4} \end{aligned}$$

Case 4. (3) applies to  $a'_i$  and (4) applies to  $c'_i$

$$\begin{aligned} |c'_i - a'_i| &= \left| \frac{c_{i-1} + 2c_i + c_{i+1}}{4} - \frac{b}{2} - \frac{a_i + a_{i+1}}{4} \right| \\ &\leq \left| \frac{c_i + c_{i+1}}{4} - \frac{a_i + a_{i+1}}{4} \right| + \left| \frac{c_{i-1} + c_i}{4} - \frac{b}{2} \right| \\ &\leq \left| \frac{a_i + a_{i+1}}{4} - \frac{c_i + c_{i+1}}{4} \right| + \left| \frac{c_{i-1} + c_i}{4} - \frac{a_{i-1} + a_i}{4} \right| \\ &\leq \frac{|c_{i-1} - a_{i-1}|}{4} + \frac{|c_i - a_i|}{2} + \frac{|c_{i+1} - a_{i+1}|}{4} \end{aligned}$$

Case 5. (4) applies to  $a'_i$  and (4) applies to  $c'_i$

$$\begin{aligned} |c'_i - a'_i| &= \left| \frac{c_{i-1} + 2c_i + c_{i+1}}{4} - \frac{a_{i-1} + 2a_i + a_{i+1}}{4} \right| \\ &\leq \frac{|c_{i-1} - a_{i-1}|}{4} + \frac{|c_i - a_i|}{2} + \frac{|c_{i+1} - a_{i+1}|}{4} \end{aligned}$$

Case 6. (5) applies to  $a'_i$  and (5) applies to  $c'_i$

$$|c'_i - a'_i| = \left| \frac{c_{i-1} + c_i}{4} - \frac{a_{i-1} + a_i}{4} \right| \leq \frac{|c_{i-1} - a_{i-1}|}{4} + \frac{|c_i - a_i|}{4}$$

Case 7. (6) applies to  $a'_i$  and (5) applies to  $c'_i$

$$|c'_i - a'_i| = \left| \frac{c_{i-1} + c_i}{4} - \frac{b}{2} \right| \leq \left| \frac{c_{i-1} + c_i}{4} - \frac{a_{i-1} + a_i}{4} \right| \leq \frac{|c_{i-1} - a_{i-1}|}{4} + \frac{|c_i - a_i|}{4}$$

We establish (7) by adding up the inequalities obtained in Cases 1-7 across all  $i$ . Indeed, each  $i$  such that  $1 < i < N$  appears on the RHS three times (once with its predecessor as  $i + 1$ , once with its successor as  $i - 1$ , and once as  $i$ ). These three terms add up to  $|c_i - a_i|$ . In turn,  $i = 1$  appears only twice (once as  $i$  and once as  $i - 1$ ), and the two terms add up to  $|c_i - a_i|/2$ . Also,  $i = N$  appears only twice (once as  $i$  and once as  $i + 1$ ), and the two terms also add up to  $|c_i - a_i|/2$ .

Let  $a_1 \leq \dots \leq a_N$  be the initial set of actions, and let  $a_1^{(k)} \leq \dots \leq a_N^{(k)}$  stand for the actions induced in period  $k$  under the best-response dynamics. Suppose that  $a_1^{(k)}$  (or  $a_N^{(k)}$ ) were bounded away from  $c_1$  (or  $c_N$ ) by some  $\varepsilon > 0$ , independent of  $k$ , for an infinite number of  $k$ 's. For such  $k$ 's, (7) implies that

$$|c'_1 - a'_1| + \dots + |c'_N - a'_N| \leq \gamma(|c_1 - a_1| + \dots + |c_N - a_N|)$$

for some  $\gamma < 1$ , which is independent of  $k$ . For other  $k$ 's, (7) clearly implies that

$$|c'_1 - a'_1| + \dots + |c'_N - a'_N| \leq |c_1 - a_1| + \dots + |c_N - a_N|.$$

This would mean that the  $L^1$ -distance between  $(a_1^{(k)}, \dots, a_N^{(k)})$  and  $(c_1, \dots, c_N)$  never decreases as  $k$  increases, but shrinks by a factor of  $\gamma$  for an infinite number of  $k$ 's. This would contradict the assumption that  $a_1^{(k)}$  (or  $a_N^{(k)}$ ) is bounded away from  $c_1$  (or  $c_N$ ) by some  $\varepsilon > 0$ , independent of  $k$ , for an infinite number of  $k$ 's. Thus,  $a_1^{(k)} \rightarrow_k c_1$  ( $a_N^{(k)} \rightarrow_k c_N$ ). But this could not happen without  $a_2^{(k)}$  ( $a_{N-1}^{(k)}$ ) converging to  $c_2$  (to  $c_{N-1}$ ). So,  $a_2^{(k)} \rightarrow_k c_2$  ( $a_{N-1}^{(k)} \rightarrow_k c_{N-1}$ ). By induction,  $a_i^{(k)} \rightarrow_k c_i$  for all  $i$ .

Suppose now that  $N > N^*$ . Let  $c_1 = \dots = c_{N-N^*} = 0$  and  $c_{N-N^*+1} \leq \dots \leq c_N$  be the equilibrium actions in the  $N^*$ -action equilibrium. By Proposition 2 in Chen, Kartik and Sobel (2008), I still have that  $c'_i = c_i$  for  $i = 1, \dots, N$ . Then, I show that  $(a_1^{(k)}, \dots, a_N^{(k)}) \rightarrow_k (c_1, \dots, c_N)$  in  $L^1$ -norm in the same way as I have shown this for  $N \leq N^*$ .

## 6 Proof of Lemma 1

Denote by  $\Delta_l$  the difference between the incompatibility in period  $k$  and the incompatibility in period  $k-1$ , when I count only coordinates up to the  $l$ -th one. In particular,  $\Delta_1$  is equal to zero, because coordinates 1 and 2 are the first pair that counts toward incompatibility, and  $\Delta_n$  is the difference between the incompatibility in period  $k$  and the incompatibility in period  $k-1$ , computed by counting all coordinates. I will establish our result by induction with respect to  $l$ .

To do so, I distinguish the following five configurations of three neighbor coordinates:

- (1)  $x_i^{k-1} < x_i^k > x_i^{k+1}$  (or analogous  $x_i^{k-1} > x_i^k < x_i^{k+1}$ );
- (2)  $x_i^{k-1} < x_i^k = x_i^{k+1}$  (or analogous  $x_i^{k-1} > x_i^k = x_i^{k+1}$ );
- (3)  $x_i^{k-1} < x_i^k < x_i^{k+1}$  (or analogous  $x_i^{k-1} > x_i^k > x_i^{k+1}$ );
- (4)  $x_i^{k-1} = x_i^k > x_i^{k+1}$  (or analogous  $x_i^{k-1} = x_i^k < x_i^{k+1}$ );
- (5)  $x_i^{k-1} = x_i^k = x_i^{k+1}$ .

We will consider these configurations in turn, and I will simultaneously show that for  $l > 1$ :

- (I)  $\Delta_l \leq -1$  if we are in Case (1);
- (II)  $\Delta_l \leq -1/2$  if we are in Case (2);



- (III)  $\Delta_l \leq 0$  if we are in Case (3);
- (IV)  $\Delta_l \leq -1/2$  if we are in Case (4);
- (V)  $\Delta_l \leq 0$  if we are in Case (5).

Performing our inductive reasoning, I will be assuming that in Cases (1)-(3),

$$(\star) \ t_l(x_l^{k-1}, x_{l+1}^{k-1}) \leq t_l(x_l^k, x_{l+1}^k) \text{ (or analogously, } t_l(x_l^{k-1}, x_{l+1}^{k-1}) \geq t_l(x_l^k, x_{l+1}^k)),$$

and in Cases (4)-(5), I will be assuming that

$$(\star\star) \text{ if } x_l^k \geq x_l^{k+1}, \text{ then } t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) \geq t_{l-1}(x_{l-1}^k, x_l^k), \text{ and if } x_l^k \leq x_l^{k+1}, \text{ then } t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) \leq t_{l-1}(x_{l-1}^k, x_l^k).$$

We will of course be showing that condition  $(\star)$  and  $(\star\star)$  can be assumed for  $l-1$ . Notice that  $(\star)$  for  $l=n$  is satisfied, because  $x_{n+1} \equiv 1$  and  $t_n$  increases with  $x_n$ . Similarly,  $(\star\star)$  for  $l=n$  is satisfied, because  $t_n(x_n^{k-1}, x_{n+1}^{k-1}) = t_n(x_n^k, x_{n+1}^k)$  in Cases (4)-(5), and  $f_n$  increases with  $t_{n-1}$ .

**Case (1).** Since  $t_l(x_l^{k-1}, x_{l+1}^{k-1}) \leq t_l(x_l^k, x_{l+1}^k)$ , it must be that  $t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) > t_{l-1}(x_{l-1}^k, x_l^k)$  because  $x_l^k > x_l^{k+1}$  and  $f_l$  strictly increases with  $t_{l-1}$  and  $t_l$ . This in turn implies that  $x_{l-1}^k < x_{l-1}^{k-1}$ , because  $t_{l-1}$  increases in both arguments. This yields  $(\star)$  for  $l-1$ .

If  $x_{l-1}^{k+1} < x_{l-1}^k$ , then  $\Delta_l = \Delta_{l-1} - 1$ , and we are in Case (3) at  $l-1$ . So, (I) for  $l$  follows from (III) for  $l-1$ . If  $x_{l-1}^{k+1} = x_{l-1}^k$ , then  $\Delta_l = \Delta_{l-1} - 1/2$ , and we are in Case (2) at  $l-1$ . In addition, it must be that  $l-1 > 1$ , because otherwise  $t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) > t_{l-1}(x_{l-1}^k, x_l^k)$  would imply that  $x_{l-1}^{k+1} < x_{l-1}^k$ . So, (I) for  $l$  follows from (II) for  $l-1$ . If  $x_{l-1}^{k+1} > x_{l-1}^k$ , then  $\Delta_l = \Delta_{l-1}$ , and we are back in Case (1) at  $l-1$ . In addition,  $l-1 > 1$  by the same argument as in the previous case.

By repeating this reasoning, I obtain that the following three cases are possible: (i)  $\Delta_l = \Delta_{l'} - 1$ , and we are in Case (3) at some  $l' < l$ , and then (I) for  $l$  follows from (III) for  $l'$ ; (ii)  $\Delta_l = \Delta_{l'} - 1/2$ , and we are in Case (2) at some  $l' < l$ , and then (I) for  $l$  follows from (II) for  $l'$ ; or (iii)  $\Delta_l = \Delta_1$ , and we are in Case (1) when coordinate 1 is reached. However, this last case is impossible, because  $(\star)$  would imply that  $x_1^k \leq x_1^{k+1}$  if  $x_1^{k-1} < x_1^k$  or  $x_1^k \geq x_1^{k+1}$  if  $x_1^{k-1} > x_1^k$ .

**Case (2).** As in Case (1), since  $t_l(x_l^{k-1}, x_{l+1}^{k-1}) \leq t_l(x_l^k, x_{l+1}^k)$ , it must be that:  $t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) > t_{l-1}(x_{l-1}^k, x_l^k)$  and  $x_{l-1}^k < x_{l-1}^{k-1}$ . This yields  $(\star)$  for  $l-1$ . If  $x_{l-1}^{k+1} > x_{l-1}^k$ , then  $\Delta_l = \Delta_{l-1} - 1/2$ , and we are in Case (1) at  $l-1$ . So, (II) for  $l$  follows from (I) for  $l-1$ . If  $x_{l-1}^{k+1} = x_{l-1}^k$ , then  $\Delta_l = \Delta_{l-1} - 1$ , and we are back in Case (2) at  $l-1$ . So, (II) for  $l$  follows from (II) for  $l-1$ . If  $x_{l-1}^{k+1} < x_{l-1}^k$ , then  $\Delta_l = \Delta_{l-1} - 1/2$ , and we are in Case (3) at  $l-1$ . So, (II) for  $l$  follows from (III) for  $l-1$ .

**Case (3).** If  $x_{l-1}^{k-1} > x_{l-1}^k$ ,  $x_{l-2}^{k-1} < x_{l-2}^k$ , and so on, that is, the direction of the inequality between  $x_m^{k-1}$  and  $x_m^k$  alternates for  $m = 1, \dots, l$ , then the incompatibility in period  $k-1$  (when counting only pairs from 1 and 2 to  $l-1$  and  $l$ ) is the highest possible, so  $\Delta_l \leq 0$ . Thus, suppose that this is not true, and take the

smallest  $m \leq l$  such that  $x_m^{k-1} < x_m^k$  and  $x_{m-1}^{k-1} \leq x_{m-1}^k$  or  $x_m^{k-1} > x_m^k$  and  $x_{m-1}^{k-1} \geq x_{m-1}^k$ . I will explore only the former case. The latter case is analogous.

Suppose first that  $x_{m-1}^{k-1} < x_{m-1}^k$ . Then, (★) is satisfied (for  $m-1$  in the place of  $l$ ) by the monotonicity of  $t_{m-1}$ . There are three cases: (i) If  $x_{m-1}^{k+1} < x_{m-1}^k$ , then we are in Case (1) at  $m-1$ . Since the incompatibility in period  $k$  among pairs from  $m-1$  and  $m$  to  $l-1$  and  $l$  can be higher only by 1 than the incompatibility in period  $k-1$  among these pairs, (III) for  $l$  follows from (I) for  $m-1$ . (ii) If  $x_{m-1}^{k+1} = x_{m-1}^k$ , then we are in Case (2) at  $m-1$ . Since in this case the incompatibility in period  $k$  among pairs from  $m-1$  and  $m$  to  $l-1$  and  $l$  can be higher only by  $1/2$  than the incompatibility in period  $k-1$  among these pairs, (III) for  $l$  follows from (II) for  $m-1$ . (iii) If  $x_{m-1}^{k+1} > x_{m-1}^k$ , then we are in Case (3) at  $m-1$ . Since in this case the incompatibility in period  $k$  among pairs from  $m-1$  and  $m$  to  $l-1$  and  $l$  cannot be higher than the incompatibility in period  $k-1$  among these pairs, (III) for  $l$  follows from (III) for  $m-1$ .

Suppose now that  $x_{m-1}^{k-1} = x_{m-1}^k$ . There are again three cases: (i) If  $x_{m-1}^{k+1} < x_{m-1}^k$ , then  $x_{m-2}^k < x_{m-2}^{k-1}$ . Indeed, otherwise  $x_{m-2}^k \geq x_{m-2}^{k-1}$  and  $x_{m-1}^{k-1} = x_{m-1}^k$  would imply that  $t_{m-2}(x_{m-2}^{k-1}, x_{m-1}^{k-1}) \leq t_{m-2}(x_{m-2}^k, x_{m-1}^k)$ , while  $x_{m-1}^{k-1} = x_{m-1}^k$  and  $x_{m-1}^{k-1} < x_m^k$  would imply that  $t_{m-1}(x_{m-1}^{k-1}, x_{m-1}^k) < t_{m-1}(x_{m-1}^k, x_m^k)$ . So, I would have that  $x_{m-1}^{k+1} \leq x_{m-1}^k$ . The inequalities  $x_{m-2}^k < x_{m-2}^{k-1}$  and  $x_{m-1}^k = x_{m-1}^{k-1}$  imply (★) in which  $l$  is replaced with  $m-2$  by the monotonicity of  $t_{m-2}$ . If in addition  $x_{m-2}^{k+1} < x_{m-2}^k$ , then we are in Case (3) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  cannot be higher than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (III) for  $m-2$ . If in addition  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in Case (2) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  cannot be higher by more than  $1/2$  than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (II) for  $m-2$ . Finally if in addition  $x_{m-2}^{k+1} > x_{m-2}^k$ , then we are in Case (1) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  cannot be higher by more than 1 than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (I) for  $m-2$ .

(ii) Suppose now that  $x_{m-1}^{k+1} = x_{m-1}^k$ . If in addition  $x_{m-2}^{k-1} = x_{m-2}^k$  and  $x_{m-2}^{k+1} \neq x_{m-2}^k$ , then we are in case (4) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  cannot be higher by more than  $1/2$  than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (IV) for  $m-2$ . If in addition  $x_{m-2}^{k-1} = x_{m-2}^k$  and  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in case (5) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  cannot be higher than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (V) for  $m-2$ .

If  $x_{m-2}^{k-1} < x_{m-2}^k$ , then (★) is satisfied (for  $m-2$  in the place of  $l$ ), since  $t_{m-2}$  increases in  $x_{m-2}$ . If in addition  $x_{m-2}^{k+1} > x_{m-2}^k$ , then we are in case (3) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  is cannot be higher than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (III) for  $m-2$ . If in addition  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in case (2) at  $m-2$ . And the incompatibility in period  $k$  among pairs from  $m-2$  and  $m-1$  to  $l-1$  and  $l$  is lower by at least  $1/2$  than the incompatibility in period  $k-1$  among these pairs. So, (III) for  $l$  follows from (II) for

$m - 2$ . If in addition  $x_{m-2}^{k+1} < x_{m-2}^k$ , then we are in case (1) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  cannot be higher than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (I) for  $m - 2$ .

Finally, if  $x_{m-2}^{k-1} > x_{m-2}^k$ , then  $(\star)$  is satisfied (for  $m - 2$  in the place of  $l$ ), again since  $t_{m-2}$  increases in  $x_{m-2}$ . If in addition  $x_{m-2}^{k+1} > x_{m-2}^k$ , then we are in case (1) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  cannot be higher than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (I) for  $m - 2$ . If in addition  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in case (2) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  is lower by at least  $1/2$  than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (II) for  $m - 2$ . If in addition  $x_{m-2}^{k+1} < x_{m-2}^k$ , then we are in case (3) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  cannot be higher than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (III) for  $m - 2$ .

(iii) Suppose now that  $x_{m-1}^{k+1} > x_{m-1}^k$ . If in addition  $x_{m-2}^{k-1} = x_{m-2}^k$  and  $x_{m-2}^{k+1} \neq x_{m-2}^k$ , then we are in case (4) at  $m - 2$ . In particular,  $(\star\star)$  is satisfied (for  $m - 2$  in the place of  $l$ ), because  $t_{m-2}(x_{m-2}^{k-1}, x_{m-1}^{k-1}) = t_{m-2}(x_{m-2}^k, x_{m-1}^k)$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  is higher when  $x_{m-2}^{k+1} < x_{m-2}^k$  than when  $x_{m-2}^{k+1} > x_{m-2}^k$ , but even in this case it cannot be higher by more than  $1/2$  than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (IV) for  $m - 2$ . If in addition  $x_{m-2}^{k-1} = x_{m-2}^k$  and  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in case (5) at  $m - 2$ . Condition  $(\star\star)$  (for  $m - 2$  in the place of  $l$ ) follows again from  $t_{m-2}(x_{m-2}^{k-1}, x_{m-1}^{k-1}) = t_{m-2}(x_{m-2}^k, x_{m-1}^k)$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  cannot be higher than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (V) for  $m - 2$ .

If  $x_{m-2}^{k-1} < x_{m-2}^k$ , then  $(\star)$  is satisfied (for  $m - 2$  in the place of  $l$ ) by the monotonicity of  $t_{m-2}$  in  $x_{m-2}$ . If in addition  $x_{m-2}^{k+1} > x_{m-2}^k$ , then we are in case (3) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  is lower by at least 1 than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (III) for  $m - 2$ . If in addition  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in case (2) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  is lower by at least  $1/2$  than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (II) for  $m - 2$ . If in addition  $x_{m-2}^{k+1} < x_{m-2}^k$ , then we are in case (1) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  cannot be higher than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (I) for  $m - 2$ .

Finally, if  $x_{m-2}^{k-1} > x_{m-2}^k$ , then  $(\star)$  is satisfied (for  $m - 2$  in the place of  $l$ ) by an analogous argument to that from the previous paragraph. If in addition  $x_{m-2}^{k+1} > x_{m-2}^k$ , then we are in case (1) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  is lower by at least 1 than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (I) for  $m - 2$ . If in addition  $x_{m-2}^{k+1} = x_{m-2}^k$ , then we are in case (2) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  is lower by at least  $1/2$  than the incompatibility in period  $k - 1$  among these pairs.

So, (III) for  $l$  follows from (II) for  $m - 2$ . If in addition  $x_{m-2}^{k+1} < x_{m-2}^k$ , then we are in case (3) at  $m - 2$ . And the incompatibility in period  $k$  among pairs from  $m - 2$  and  $m - 1$  to  $l - 1$  and  $l$  cannot be higher than the incompatibility in period  $k - 1$  among these pairs. So, (III) for  $l$  follows from (III) for  $m - 2$ .

**Case (4).** By (★★),  $t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) > t_{l-1}(x_{l-1}^k, x_l^k)$ . So,  $x_{l-1}^k < x_{l-1}^{k-1}$ , since  $t_{l-1}$  is increasing in  $x_{l-1}$ . This implies (★) for  $l - 1$ . If in addition  $x_{l-1}^{k+1} < x_{l-1}^k$ , we are in Case (3) at  $l - 1$ . Since coordinates  $l - 1$  and  $l$  are half-incompatible in period  $k - 1$ , and are compatible in period  $k$ , (IV) for  $l$  follows from (III) for  $l - 1$ . If in addition  $x_{l-1}^{k+1} > x_{l-1}^k$ , we are in Case (1) at  $l - 1$ . In addition, it must be that  $l - 1 > 1$ , because otherwise (★) would imply that  $x_{l-1}^{k+1} < x_{l-1}^k$ . Since coordinates  $l - 1$  and  $l$  are incompatible in period  $k$ , and are half-incompatible in period  $k - 1$ , (IV) for  $l$  follows from (I) for  $l - 1$ . If in addition  $x_{l-1}^{k+1} = x_{l-1}^k$ , we are in Case (2) at  $l - 1$ . As previously, it must be that  $l - 1 > 1$ . Since coordinates  $l - 1$  and  $l$  are half-incompatible, both in period  $k - 1$  and in period  $k$ , (IV) for  $l$  follows from (II) for  $l - 1$ .

**Case (5).** By (★★),  $t_{l-1}(x_{l-1}^{k-1}, x_l^{k-1}) = t_{l-1}(x_{l-1}^k, x_l^k)$ . So,  $x_{l-1}^k = x_{l-1}^{k-1}$ , since  $t_{l-1}$  is increasing in  $x_{l-1}$ . If in addition  $x_{l-1}^{k+1} < x_{l-1}^k$ , then  $t_{l-2}(x_{l-2}^{k-1}, x_{l-1}^{k-1}) > t_{l-2}(x_{l-2}^k, x_{l-1}^k)$ , because  $f_{l-1}$  strictly increases with  $t_{l-2}$ . Thus, (★★) is satisfied for  $l - 1$ , and so we are in Case (4) at  $l - 1$ . Since coordinates  $l - 1$  and  $l$  are half-incompatible, in period  $k$  and compatible in period  $k - 1$ , (V) for  $l$  follows from (IV) for  $l - 1$ . The arguments are analogous when  $x_{l-1}^{k+1} > x_{l-1}^k$ . So, suppose that  $x_{l-1}^{k+1} = x_{l-1}^k$ . Then, it must be that  $t_{l-2}(x_{l-2}^{k-1}, x_{l-1}^{k-1}) = t_{l-2}(x_{l-2}^k, x_{l-1}^k)$ , because  $f_{l-1}$  strictly increases with  $t_{l-2}$ . Thus, (★★) is satisfied for  $l - 1$ , and (V) for  $l$  follows from (V) for  $l - 1$ .

## 7 References

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