Optimal Stopping With Regret
When to Stop if Nature is malevolent

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Abstract

This paper studies an optimal stopping problem in which the decision maker is subject to anticipated regret. Once the decision maker stopped she observes future outcomes. Regret arises, because the decision maker evaluates her outcome against the ideal future outcome. I propose a new way to model regret over the future and show that it considerably affects the decision maker’s behavior. I fully characterize the solution of the model and show that stopping occurs later than in a model without regret. However, not in a monotone way. I also provide comparative statics for the main parameters. More generally I view this model as a game against nature and provide an application to political economy.

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1 Introduction

Consumers often regret their decisions. It is therefore an important economic question to understand how regret influences consumers’ behavior. For example, regret is one possible explanation for why retailers offer price guarantees. In this paper I study the effects of regret on a decision maker’s behavior in the framework of an optimal stopping problem.

Imagine you want to buy a camera at an online marketplace like Ebay or Amazon. For simplicity let us assume that you have already decided which model you want to buy. You are unsure, however, about when to buy, because the price evolves over time. This is a classical optimal stopping problem. Now suppose that after you have bought the camera, you receive an email from Amazon telling you that your camera has just gone on sale. Do you regret having bought prematurely?

Surprisingly regret over the future has received scant attention in the setting of optimal stopping problems. Viefers and Strack (2014) find in their model that regret over the future does not have any role to play. To the best of my knowledge, Hayashi (2009) is the only author who provides a framework in which it does. In his model regret arises because the decision maker has some fundamental uncertainty about which model of the world is the correct model.

Here I propose a different approach. I allow the time until when future outcomes are observed to enter into the decision maker’s preferences. I assume that the decision maker is uncertain about this time and that the uncertainty is only resolved once the decision maker has stopped. If the time is large, it is likely that a high outcome will be attained in the future and regret is high.

Crucially I allow the time at which the observation stops to depend on the future path of outcomes (it is a stopping time). For example, one such time could prescribe to stop the first time a future outcome reaches a level of fifty dollars, say. Thus for how long the observation of future outcomes
actually does go on, depends on how outcomes evolve and in particular which path is realized.

It follows that the timespan during which outcomes are tracked, depends on the outcome at which the decision maker stops. For example, if she stops at an outcome of fifty the timespan is zero. While if she stops at an outcome of a hundred, the timespan will be considerable. As this affects the decision maker’s regret, she takes this into account when she makes her decision. Therefore taking the time of observation as a stopping time guarantees that the decision maker’s stopping decision depends on this time. This is the key mechanism why regret over the future plays a role in this model.¹

I assume that once the decision maker has stopped, tracking future outcomes becomes costly. I model this by introducing linear search costs. These costs can be interpreted as opportunity costs or as linear discounting. This mirrors the observation that while regret does matter for individuals, so does the time when it accrues. If many future outcomes have to be observed for regret to materialize, the negative emotion ought to be small compared to a situation in which one is immediately confronted with a better outcome than the one obtained. If Amazon sends an email with new price information tomorrow, this will differentially affect how the decision maker feels compared to when it sends it in a year’s time.

I find, in line with Hayashi (2009), that regret induces decision makers to delay their decisions. This offers one possible explanation for observed phenomena like price guarantees or the recent popularity of applications that offer recommendations of whether to buy now or wait longer. All of these mechanisms mitigate the regret the consumers may experience and so avoid that they delay their decisions. In general, when consumers have regret considerations a volatile price setting policy may be harmful.

That decision makers delay their decision when they experience regret is intuitive. Regret is a cost and can be reduced by stopping at a better outcome.

¹Note that the discounting of the decision maker is the second channel.
price. However, surprisingly, the amount by which the decision maker delays her decision is not always monotone in the search costs. One would expect otherwise, it is striking that the decision maker settles with a low outcome although search costs are low and potential regret is high. There is a number of other results which I obtain from my model: I show that the decision maker exerts self control, in a sense to be made precise. Moreover, the decision maker may delay her decision forever, a result obtained in O’Donoghue and Rabin (2001) in a model of hyperbolic discounting.

More formally, in the model the best of all outcomes (the lowest price) up to date is recorded in a reference point. Its first value is an important parameter of the model. I interpret it as the decision maker’s ex-ante idea about what she expects to get from the problem. Over time, the reference point evolves. The decision maker’s regret stems from the comparison of her outcome to the value of this reference point.

To capture the decision maker’s preferences it is natural to use a modeling device and to introduce a second player into the model. First the decision maker stops, then the second player is drawn. Its defining characteristic is the time until which it observes future outcomes, which depends on its search costs. As such the second player has a natural interpretation: I think of it as an alter ego of the decision maker.

It seems problematic to assume that the decision maker is able to form a prior belief about which of her alter egos she is going to be confronted with. I therefore adopt a distribution free approach, the minimax regret decision criterion, see Savage (1951). The decision maker acts as if she would face the worst of her alter egos, I call it the evil self. The evil self is the one alter ego, whose stopping time maximizes the decision maker’s regret.

The solution to my model differs in important ways from the one of a model without regret. In a stopping problem without regret, the optimal strategy is to specify a cutoff: a reservation value at which one stops, the first time this value is attained. In the model with regret, the stopping problem
is two dimensional. I show that the decision maker’s optimal stopping rule is still a cutoff rule. But the cutoff crucially depends on the reference point. Instead by a single point, the stopping region is now characterized by a function. This leads to an interesting interaction between the decision maker and her evil self.

I provide a full characterization of the evil self’s and the decision maker’s stopping problem and show that regret over the future does matter in this setting. This result is driven by the fact that the evil self chooses a stopping time and that the decision maker is impatient.

The fact that regret induces the decision maker to stop later is one of the main findings of the model. This implies that the optimal cutoff is strictly higher than in a model without regret. An interesting example of how companies try to mitigate regret considerations, is provided by the website kayak.com. This website not only offers a price tracker, but gives a recommendation of whether one should buy now or wait longer.\(^2\) In the context of regret, this recommendation seems to have a very specific purpose: to increase the consumer’s confidence in her decision, thus avoid that she tracks prices and delays her decision.

I also find that regret may induce a behavior which can be interpreted as procrastination. When the decision maker is relatively receptive to regret considerations, or the evil self’s search costs are low, which implies that outcomes are tracked for long and thus regret looms large, she begins to increasingly postpone her decision. In the limit, she does so indefinitely and puts off the decision forever. Although this implies that she will never receive a reward from her stopping decision, it also implies that she will not regret. In a situation where regret dominates, this is the optimal strategy. This is

\(^2\)The precise way in which this assessment is reached is a well kept business secret. The company is only willing to disclose that its algorithm is based on demand data from the past, and data on the consumers’ behavior. Nate Silver from the blog fivethirtyeight, see http://fivethirtyeight.com/features/when-to-hold-out-for-a-low-airfare/, casually looked at a small sample of recommendations and found them to be mostly right.
feasible because of the model’s sequential structure. The decision maker can prevent her evil self from ever being able to move.

Another important result is that the decision maker exhibits self control in this model. The decision maker moves first, so she can influence the evil self’s outcome via the values at which she stops. Whether she chooses to control her evil self or not depends on the initial value of the reference point. When the reference point is not too high, the evil self attempts to increase regret and the decision maker has control over it. I show that this induces the decision maker to choose smaller cutoffs, because this hurts the evil self. In her attempt to control her evil self, the decision maker sacrifices some of her payoff from stopping in order to worsen the evil self’s situation and thus reduce regret.

I also address the question of comparative statics. I am particularly interested in how the discount rate of the decision maker and the evil self’s search cost change the solution. I find that an increase in the decision maker’s impatience leads to earlier stopping decisions. This is analogous to the result in an optimal stopping problem without regret. However, while in the standard model there is no constraint on how small the stopping time can be chosen, this is not the case in this framework. In my model, the decision maker has at least to continue until her payoff is non-negative. This bounds the time of continuation from below.

With respect to the evil self’s search costs, I find an interesting non-monotonicity. I show that the amount of time for which the decision maker postpones her decisions is not monotone in the costs. This stems from the interplay between a desire for self control and the need to continue for longer to offset the cost from regret. Surprisingly this non-monotonicity may also arise when the drift of the underlying process changes. Although the game becomes more favorable when the drift increases, it can happen that the decision maker stops earlier due to the fact that also her evil self profits from this change. This shows that the non-monotonicity is not confined to search
costs, it is deeply embedded in the model.

Stopping problems, where the decision maker is subject to regret, have received relatively little attention. Hayashi (2009) analyses a stopping problem in which regret over having stopped prematurely can play a role. While in my model the stochastic framework is known, in Hayashi (2009) the decision maker learns about it from each new realization via Bayes’ rule. The decision maker, however, also believes that with a certain probability she uses the wrong model specification and this distorts the posterior belief(s). Time is discrete and an optimal policy specifies a cutoff for every time period. For the case of a decision maker who commits to one such policy, regret over the future does matter, but only if posterior beliefs are distorted. If the decision maker is confident that she uses the right model, regret over the future has no role to play. In my model, by contrast, the decision maker is perfectly informed about the stochastic framework. For a special case Hayashi (2009) demonstrates that more uncertainty about the true model induces the decision maker to stop later. This parallels the result I obtain. My model offers an alternative explanation for why regret leads to more aggressive behavior.

Viefers and Strack (2014) provide a model which is closest to this paper’s. The authors demonstrate that in their setting regret over the future has no role to play. The decision maker’s behavior is observationally equivalent to an expected utility maximizer’s.

The logic behind this result is that in their model the best value in the future is independent of the value at which the decision maker stops. As a result regret is the same in expectation at any of the decision maker’s stopping times and as it is constant, it does not affect the optimal behavior. In my model this is no longer true, because the time at which the evil self stops is not independent of the underlying stochastic process. Therefore regret over the future is not constant and does influence the decision maker’s stopping decision.

The empirical literature on regret is large. Connolly and Zeelenberg
(2002) provide a survey about the empirical results related to regret. Zee-
lenberg (1999) studies how the anticipation of regret impacts on the de-
cision makers’ behavior. He finds that decision makers can become both
risk-seeking or risk-averse, depending on which of the choices is the regret
minimizing one. In my model, stopping occurs later, which can be viewed as
willingness to accept a higher risk. The article of Cooke et al. (2001) is most
related to this paper. The authors conducted a lab experiment, in which
the subjects had to make a purchase decision. After they had made their
buying decision, subjects received post-purchase price information. Cooke
et al. (2001) find that regret over the future severely affects their subjects.
Moreover it seems to be more important than regret over past decisions.

In the finance literature, other authors also incorporated regret. For ex-
ample Pye (1971) finds that the common practice of dollar averaging observed
in financial markets is related to hedging against large regrets. Moreover,
while minimax regret rationalizes dollar averaging, expected utility maxi-
mization does not. Bawa (1973) considers a similar setting as Pye (1971). In
a discrete time setting with finite time horizon and for an arithmetic random
walk, she derives the optimal minmax regret policy for selling an asset. In
both of these papers regret is over the past.

The evil self’s problem is technically a problem of optimally stopping the
maximum process associated with some stochastic process. This literature
originated in the pricing of exotic options, see for example Guo and Shepp

Finally the model relates to the literature on blame games (see e.g. Anand
(1998), Gilmour (2003), and DeScioli and Bokemper (2014)). In Section six
I offer a reinterpretation of the model in the context of political economy
and show that the model provides a mechanism for the explanation of some
salient facts in politics. For example, the presence of a critical press may
improve the outcome a politician chooses.
Structure of the paper. The paper is structured as follows. Section two sketches the solution of the baseline model, in which the decision maker is not subject to regret. Section three looks at the model when regret considerations are present. I prove that the full problem can be decomposed into two nested problems which can be solved sequentially. In Section four I solve the first of these problems, the evil self’s stopping problem. In Section five I derive the solution to the decision maker’s problem. Section six analyses the comparative statics of the model. The focus lies on how the decision maker’s discount rate, the evil self’s search cost and the drift affect the solution. Section seven reinterprets the model and discusses an application to political economy. Section eight concludes.

2 The Model without Regret

In this Section I introduce the optimal stopping model when there is no evil self. In this case the decision maker does not suffer from regret. The solution to this problem is standard.

Consider a decision maker who can choose a time at which to stop. I assume that outcomes evolve randomly over time and, for simplicity, identify the decision maker’s utility from the decision directly with its outcome. Throughout the paper the process driving the outcomes will be a Brownian motion with drift. The decision maker is impatient and discounts the future at rate $r > 0$. In this setup, the decision problem can be formulated as a standard stopping problem.

In this class of problems the decision maker has to reconcile two opposing forces: on the one hand continuation is costly, because there is discounting. On the other hand continuation may lead to better outcomes. The optimal stopping rule balancing these two forces is of a particular type: it is a cutoff rule. There exists a threshold $s^*$ such that if the value of the outcome is below this threshold, the decision maker continues. And she stops at the
first time the outcome weakly exceeds $s^*$.

It is easy to see why this policy is optimal. Note first that the increments of a Brownian motion are independent of the current state. Thus the probability of an improvement is the same at a low and at a high current outcome. If, however, the outcome is low, discounting is relatively less severe and continuing with the gamble is profitable. When the value is high, discounting has bite and thus it is best to stop immediately.

Formally, let $(X_t)_{t \geq 0}$ be a Brownian motion with drift $\mu$, started from $x \in \mathbb{R}$, and variance $\sigma^2$. Let $P_x$ be the measure under which the process $(X_t)_{t \geq 0}$ starts at $x$ and assume that $x \mapsto P_x$ is measurable for all $x$. Under measure $P_x$, $X_t$ is a Brownian motion which starts at the initial value $x$. The decision maker chooses a stopping time $\tau$ such as to maximize her expected discounted payoff given by

$$E_x e^{-\tau r} X_\tau,$$

where $E_x$ is the expectation operator associated with measure $P_x$. Standard arguments show that $s^* = 1/b_1$, where $b_1 \equiv (-\mu + \sqrt{\mu^2 + 2r\sigma^2}) / \sigma^2$. It is optimal to stop the first time $X_t$ exceeds $s^*$. The value equals

$$W(x) = \begin{cases} e^{b_1 (x-s^*)} s^* & \text{if } x < s^* \\ x & \text{otherwise} \end{cases}.$$  

Observe that the decision maker continues only for small initial values of the Brownian motion. Whenever $x \geq s^*$ it is best to stop immediately. Thus, the decision maker quits the game and takes the money home.

As we will see in the next Subsection, the introduction of regret changes the optimal behavior of the decision maker considerably.

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3See for example Dixit (1993) for details.
3 The Model with Regret

In this Section I introduce a second player into the standard stopping problem. I first set up the model and then discuss the underlying behavioral assumptions, which motivate the introduction of a second player. One important result of this Section is that the problem with a second player can be solved in a sequential way. One may start with the second player’s problem and given its solution, proceed with the first player’s problem.

In the context of regret I view the second player as another self of the decision maker. Suppose the stopping action was taken. I imagine our decision maker as being tempted to compare his outcome to a reference outcome, which evolves over time. After she has stopped, she gets to see the realization of future outcomes. The reference point is taken to be equal to the best of these future outcomes. The first value of the reference point is a parameter of the stopping problem. The decision maker experiences regret, whenever the outcome she attained is less then the reference outcome. In the literature this is called regret over the future. The decision maker’s other self determines the timespan during which future outcomes are observed and thus the amount of regret. I adopt a minmax regret approach. One advantage of this approach is that it is distribution free. As such it does not require the decision maker to form priors about which of her other selves she is going to face. The decision maker is thus confronted with an evil self. Its task is to maximize regret, that is to inflict as much harm on the decision maker as possible. Before proceeding with the discussion, I formally introduce the model.

Let \((X_t)_{t \geq 0}\) be the Brownian motion introduced in the previous Section. Denote by \(S_t\) the maximum process of the Brownian motion. To allow this process to start from any positive initial value \(s\), set \(S_t = s \vee \max_{0 \leq u \leq t} X_u\). \(s\) is the initial value of the reference point. Let \(P_{x,s}\) be the measure under which the process \((X_t, S_t)_{t \geq 0}\) starts at \((x, s)\) and assume that \((x, s) \mapsto P_{x,s}\) is measurable for all \((x, s)\). Let \(E\) be the state space of the strong Markov
process \((X_t, S_t)\). The Brownian motion is allowed to take values in all of the real numbers, while the maximum process is assumed to be non-negative throughout, that is \(s \geq 0\). The state space \(E\) of the process is thus \(E = \{(x, s) \in \mathbb{R} \times \mathbb{R}_+ : x \leq s\}\).

I next introduce regret into the model and start with the case where there is no search cost for the evil self. Commonly regret is defined as the difference between the outcome the agent obtained from the decision she took, and the best of all available outcomes. In the context of our stochastic framework, if the agent stops at time \(\tau\) her undiscounted payoff is \(X_\tau\) and if her alter ego stops at time \(T > \tau\) the best of all outcomes is recorded in the value of the running maximum at time \(T, S_T\). Thus the value of the reference point at time \(T\) is \(S_T\). Consequently, the regret the agent experiences from stopping at time \(\tau\) and feeling tempted to follow the problem until time \(T\) is \(X_\tau - S_T\).

Assume that after the decision maker has stopped, tracking future outcomes becomes costly. Let the cost be \(c > 0\). These can be interpreted as opportunity costs, the decision maker has to divert attention from the current, more relevant problem, or as linear discounting. Suppose that the decision maker tracks outcomes until some time \(T\). This time affects her regret and hence is a parameter in the decision maker’s utility function. Ex ante the decision maker is uncertain about which time \(T\) will be realized. But this suggests the modeling device introduced above. Identify each such time with an alter ego of the decision maker. So, once the decision maker has stopped, an alter ego is drawn, which tracks future outcomes until time \(T\). For this alter ego, the value of the best outcome is recorded in the running maximum, \(S_T\) and search costs are \(c(T - \tau)\). This defines regret in the dynamic setting, where the reference point evolves over time \(^4\). Finally, the

\(^4\)Other specifications are possible. For example, instead of search costs one could assume that the decision maker has limited attention and loses interest according to an exponential random variable with intensity \(\delta\), which is independent of the stochastic processes. If \(S_T\) is the value of the maximum at time \(T\), expected regret at time \(\tau\) would be \(X_\tau - E\left[e^{-\delta(T-\tau)}S_T \mid X_\tau, S_T\right]\). This specification is equivalent to a model in which regret is discounted. For the case of a GBM preliminary results suggest that the solution is
decision maker does not face any alter ego, but the worst one, his evil self. The evil self is characterized by the specific time $T$, which maximizes the decision maker’s expected regret.

**Objective.** Let $\kappa \in (0,1)$ measure the intensity of regret the decision maker is subject to. The decision maker’s value from stopping at time $\tau$, while the evil self stops at time $T$ is given by

$$\sup_{\tau \geq 0} \inf_{T \geq \tau} \mathbb{E}_{x,s} e^{-\tau \tau} \left( (1 - \kappa) X_{\tau} + \kappa (X_{\tau} - (S_T - c(T - \tau))) \right),$$

where $\mathbb{E}_{x,s}$ is the expectation operator associated with the measure $P_{x,s}$. The objective function makes the previous discussion precise. The alter ego maximizes regret by choosing $T$ such that $S_T - c(T - \tau)$, the value of the decision maker’s reference point net of sampling cost, is largest. This minimizes the value in the above expression. Thus the decision maker’s other self is an *evil self*. The decision maker in turn chooses her time in a way that picks the best among all worst regret outcomes.

The expression in (1) calls attention to the possibility that this problem has a sequential structure. And indeed, as the next Lemma shows, this intuition is correct. The Lemma makes use of the strong Markov property of the process $(X_t, S_t)$. While the Markov property goes a long way towards the desired sequential structure, a technical requirement concerning the measurability of the involved conditional expectations is needed. The continuity of the sample paths of $(X_t, S_t)$ is enough in this context. Thus the problem is indeed sequential and can be solved by backward induction.

**Lemma 1.** Let $\tau$ and $T$ be Markov times adapted to the filtration generated by $(X_t, S_t)$. The problem in (1) is equivalent to

$$\sup_{\tau \geq 0} \mathbb{E}_{x,s} e^{-\tau \tau} \left( X_{\tau} - \kappa \sup_{T \geq 0} \mathbb{E}_{X_{\tau}, S_{\tau}} (S_T - c T) \right)$$

qualitatively similar to the solution of the model in this paper.
where $\mathbb{E}_{X_t,S_t}$ is the expectation operator conditional on the process $(X_t, S_t)_{t \geq \tau}$ starting from $(X_\tau, S_\tau)$.

**Proof.** By linearity of the expectation operator, the problem in (1) is equivalent to

$$\sup_\tau \inf_{T \geq \tau} \{ \mathbb{E}_{x,s} e^{-r\tau} X_\tau - \kappa \mathbb{E}_{x,s} e^{-r\tau} (S_T - c(T - \tau)) \}.$$  

As the evil self does not optimize over $\tau$, the infimum can be brought in to give

$$\sup_\tau \left\{ \mathbb{E}_{x,s} e^{-r\tau} X_\tau - \kappa \mathbb{E}_{x,s} e^{-r\tau} (S_T - c(T - \tau)) \right\}.$$  

Since the process $S_t$ is nonnegative on $E$, $\inf(-x) = -\sup x$ and the law of iterated expectations shows that the above is equivalent to

$$\sup_\tau \left\{ \mathbb{E}_{x,s} e^{-r\tau} X_\tau - \kappa \mathbb{E}_{x,s} e^{-r\tau} (S_T - c(T - \tau)) \right\}.$$  

The function $e^{-r\tau}$ is $\mathcal{F}_T$-measurable and hence can be taken out of the inner expectation. Then define $\tilde{T} = T - \tau$ and rewrite

$$\sup_\tau \left\{ \mathbb{E}_{x,s} e^{-r\tau} X_\tau - \kappa \sup_{\tilde{T} \geq \tau} \mathbb{E}_{x,s} e^{-r\tilde{T}} [S_{\tilde{T}+\tau} - c\tilde{T}] |\mathcal{F}_\tau] \right\}.$$  

The next step is to exchange the expectation operator $\mathbb{E}_{x,s}$ and the supremum. Note that $e^{-r\tau}$ is not affected by the time $\tilde{T}$ chosen by the evil self. The process $(S_t)_{t \geq 0}$ has continuous sample paths. Thus the supremum

$$\sup_{\tilde{T} \geq 0} \mathbb{E}_{x,s} [S_{\tilde{T}+\tau} - c\tilde{T}] |\mathcal{F}_\tau]$$  

is a measurable function. This fact justifies the exchange of the expectation operator and the supremum and one arrives at

$$\sup_\tau \left\{ \mathbb{E}_{x,s} e^{-r\tau} X_\tau - \kappa \sup_{\tilde{T} \geq 0} \mathbb{E}_{x,s} [S_{\tilde{T}+\tau} - c\tilde{T}] |\mathcal{F}_\tau] \right\}.$$  

The strong Markov property of the process $(X_t, S_t)$ implies that for any
bounded and measurable function \( f \) and stopping time \( \tau \)

\[
E_{x,s} [ f (\{(X_{t+\tau}, S_{t+\tau}) : t \geq 0\}) | \mathcal{F}_\tau] = E_{X_\tau, S_\tau} f \left( \{ (\tilde{X}_t, \tilde{S}_t) : t \geq 0 \} \right)
\]

where \((\tilde{X}_t, \tilde{S}_t)\) is another Markov process started at \((X_\tau, S_\tau)\). Hence the evil self optimizes over stopping times \( \tilde{T} \geq 0 \) and a process \((\tilde{X}_t, \tilde{S}_t)\) started at \((X_\tau, S_\tau)\) and the problem may be written as

\[
\sup_{\tau} E_{x,s} e^{-r\tau} \left\{ X_\tau - \kappa \sup_{\tilde{T} \geq 0} E_{X_\tau, S_\tau} \left[ S_{\tilde{T}} - c\tilde{T} \right] \right\}.
\]

This concludes the proof of the Lemma.

I solve the evil self’s problem in the next Section. The solution to the full problem in (1) is given in Theorem 12 in Section 5.

4 The Evil Self’s Problem.

In this Section I solve the evil self’s problem. For higher clarity, I invert the order, first discuss the solution and then provide the formal derivation in the second Subsection.

4.1 Overview of the Solution

The evil self’s objective is to choose a stopping time such that the decision maker’s regret is maximized. From Section three we know that this is equivalent to choosing a stopping time \( T \geq 0 \) such that \( E_{x,s} (S_T - cT) \), is maximized. There is two opposing effects which determine whether continuation or stopping is optimal. On the one hand continuation is costly, because future observations have to be paid for. On the other hand continuation is the only way to attain higher values. The key quantity, which characterizes the solution to this problem is the gap, \( s - x \), between the current maximum
$s$ and the current value $x$. Observe that new maxima are only attained when $x = s$. Therefore, if the gap is large, the expected time to return to $x = s$, and so to improve, is long. This implies that payoff is severely reduced by search costs. The larger the gap $s - x$ becomes, the more pronounced this is and the less the evil self has to gain from continuation. On the other hand, if the evil self stops, it always has a payoff of $s$. Thus, while the payoff from continuation decreases in the size of the gap $s - x$, the payoff received upon stopping, $s$, is constant.

This suggests that for every value of the current maximum $s$, there is a value $g(s) < s$ such that it is optimal to stop the first time the Brownian motion $X_t$ falls to $g(s)$. This is true for every value of the current maximum, $S_t$, and thus defines the continuation region $C$, given by $C = \{(x, s) \in E : g(S_t) < X_t\}$.

Figure 1 shows the continuation and stopping regions in the $(x, s)$-space. The boundary $g(s)$ separates these two regions. The process $S_t$ is constant to the left of the diagonal, where $x = s$, and only increases along the diagonal. When $x = s$, new maxima are attained. The process $X_t$ takes values in the area to the left of and including the diagonal.

The Figure summarizes the discussion so far. It shows, that whenever the gap between $x$ and $s$ becomes large, that is $X_t$ is in the region to the left of $g(s)$, it is optimal to stop. On the other hand, the boundary $g(s)$ stays away from the diagonal, where $x = s$. When $x = s$, the evil self cannot stop. It is the best place to be in this framework. At the next step, either a new maximum is attained, or the Brownian motion does not move far from $x = s$. In the latter case the expected time to return to $x = s$ is small and costs are low. Therefore, the evil self never stops at $x = s$.

The value function $V(x, s)$ for the evil self’s stopping problem will depend on the initial values $(x, s)$ of the processes $X_t$ and $S_t$. These are the initial values for the evil self’s problem, but they are chosen by the decision maker. The value $V(x, s)$ will be shown to be increasing in both variables $x$ and $s$. 

Figure 1: The optimal stopping boundary $g(s)$, the continuation region and the stopping region.
Thus, if the decision maker stops at high values of $x$ she not only improves her value, but also the value of the evil self, because the gap $s - x$ becomes smaller. In Section 5, where I solve the decision maker’s problem, I show how these two opposing effects influence the solution.

4.2 The Solution to the Evil Self’s Problem

The evil self’s problem is a standard stopping problem of the discounted maximum process of a diffusion process. A similar problem has been studied in Dubins et al. (1993), but with unit variance parameter. The general method of solving these problems relies on techniques outlined in for example Guo and Shepp (2001), Graversen and Peskir (1998) and Peskir and Shiryaev (2006). Following these methods, I obtain a second order differential equation for the value function. The interpretation of this differential equation is as a no-arbitrage equation. Its constants of integration are determined by the value matching and smooth pasting conditions. However, for the optimal termination of a maximum process, these conditions are not sufficient. One needs to take a stance on the behavior of the optimal value where $X_t = S_t$. For the problem to have a well-defined value the value process must be reflected at the line $x = s$. This is the normal reflection condition. Along the line $x = s$ the agent’s value is immediately thrown back into the continuation region where $x < s$. The value matching, smooth pasting and normal reflection conditions generate another differential equation that characterizes the stopping boundary. In my framework this equation has an algebraic solution. For the general case Matomäki (2013) provides an algorithm, which can be used to solve this stopping problem numerically by viewing it as a sequence of simpler stopping problems.

**The Evil Self’s Problem.** In Lemma 1, I have shown that we can first solve the evil self’s problem and then proceed to the decision maker’s problem. Suppose the decision maker stops at time $\tau$. There the stopped
process is \((X_t, S_t)\). Without loss of generality, let \(X_t = x\) and \(S_t = s\). The evil self, then, chooses a stopping time \(T\) to maximize its expected payoff,

\[
V_*(x, s) \equiv \sup_{T \geq 0} \mathbb{E}_{x,s} (S_T - cT).
\]  

(2)

The following Lemma confirms that stopping on the line \(x = s\) is never optimal.

**Lemma 2.** It is never optimal to stop when \(X_t = S_t\).

**Proof.** See for example Peskir and Shiryaev (2006).

Since stopping at \(x = s\) is not optimal there has to exist a stopping boundary \(s \mapsto g(s)\), which for every value of the maximum process \(s\), determines a value for the Brownian motion at which to stop. If the Brownian motion wanders far from the current maximum, it is optimal to stop. The reason is that even with a positive drift, the chance of improving on the current maximum in a reasonable amount of time is small. Consequently, the continuation region \(C\) should be given by \(C = \{(x, s) \in E : X_t > g(S_t)\}\), while the stopping region is \(E/C\). It follows that the optimal stopping time \(T\) is given by \(T = \inf \{T > 0 : X_t \leq g(S_t)\}\).

To obtain a solution to (2) I follow Graversen and Peskir (1998), exploit the Markovian structure of the problem and impose the value matching and smooth pasting conditions. This gives a candidate value function \(V(x, s)\). In the proof of Theorem 3, I show that the candidate \(V(x, s)\) is equal to \(V_*(x, s)\) in (2). Since in the continuation region \(C\), the value \(V(x, s)\) only changes when the Brownian motion \(X_t\) changes, I am led to formulate the following free boundary problem:

\[
\begin{align*}
\mathbb{L}_X V(x, s) &= c, \quad g(s) < x < s \\
V(x, s)|_{x=g(s)}^+ &= s, \\
\frac{\partial V(x, s)}{\partial x}|_{x=g(s)}^+ &= 0.
\end{align*}
\]  

(3)
The first equation involves the infinitesimal generator of the process $X_t$, which is given by $L_X V(x, s) = \mu \frac{\partial}{\partial x} V(x, s) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} V(x, s)$. This equation is essentially a non-arbitrage condition. Holding the asset entitles the owner to consume the dividends, which are zero here, and the capital gains of the asset. These gains have to be equal to the return from investment into a risk-free asset with value $c$. That is stopping is not optimal as long as the game is “fair”, which it is precisely in the continuation region. This no-arbitrage condition gives rise to an ordinary second order differential equation with solution
\[
V(x, s) = -\frac{\sigma^2}{2\mu} C_1 e^{-\frac{2\mu x}{\sigma^2}} + C_2 + \frac{cx}{\mu},
\]
where $C_1$ and $C_2$ are the constants of integration.

To determine the constants of integration, consider the last two equations in (3): The first equation is the value matching condition. It says that upon stopping the value function $V$ has to equal the value received from stopping, being $s$. The second condition is the smooth pasting condition, which says that not only are the values the same at the stopping boundary, but also that the continuation value and the value received upon stopping have the same first derivative. These two conditions determine the constants of integration and I obtain
\[
V(x, s) = s + \frac{\sigma^2}{2\mu^2} c \left[ e^{\frac{2\mu}{\sigma^2}(g(s)-x)} - 1 + \frac{2\mu}{\sigma^2} (x - g(s)) \right].
\]
This value function still involves the unknown boundary $g(s)$ so an additional condition is required to arrive at a solution to this problem. This condition is known in the literature as normal reflection. Its mathematical statement is
\[
\frac{\partial V(x, s)}{\partial s} \bigg|_{x = g(s)} = 0.
\]
Every process, which is reflected at a boundary, satisfies this condition. A prime example is a Brownian motion reflected at the origin. The condition
guarantees that the reflected process is immediately “thrown back” into the interior of the state space. Reflection of the process $V(x,s)$ along the line $x = s$ ensures that the value is finite.

The normal reflection condition for $V$ gives a first order non-linear differential equation for the boundary $g(s)$:

$$g'(s) = \frac{\mu}{c} \left( 1 - e^{\frac{2\mu}{\alpha^2}(g(s) - s)} \right)^{-1}. \quad (4)$$

The solution $g(s)$ to this differential equation is given by $g(s) = s - \alpha$, where $\alpha = -\frac{\sigma^2}{2\mu} \ln (1 - \frac{\mu}{c})$. Observe that $g(s)$ has two important properties: First it does not cross the line $x = s$. And second, any other solution $\gamma(s)$ satisfies $\gamma(s) \leq g(s)$ for all $s$. Hence $g(s)$ is the maximal solution, which does not cross the line $x = s$. The following Theorem verifies that the candidate value function $V(x,s)$ equals $V_\ast(x,s)$ defined in (2).

**Theorem 3.** Let $c > \mu$:

1) The maximal solution to (4) is given by $g_\ast(s) = s - \alpha$.

2) The stopping time $\tau_{g_\ast} = \inf \{ t > 0 : X_t \leq g_\ast(S_t) \}$ is optimal for problem (2).

3) The value function $V(x,s)$ of nature’s problem (2) is given by

$$V(x,s) = \begin{cases} 
  s + \frac{\sigma^2}{2\mu^2} c \left( e^{\frac{2\mu}{\alpha^2}(g_\ast(s) - s)} - 1 + \frac{2\mu}{\alpha^2} (x - g_\ast(s)) \right), & \text{for } g_\ast(s) < x \leq s; \\
  s, & \text{for } x \leq g_\ast(s).
\end{cases}$$

If $c < \mu$, the value is unbounded.

**Proof.** See the Appendix Section 9.1. \hfill \Box

I collect a few facts relating to the value function and the maximal boundary $g_\ast(s)$ in the next Corollary. I will make intensive use of them in the proof of the Main Theorem, Theorem 12.
Corollary 4. (1) Nature’s value function is strictly increasing and convex in $x$.

(2) Let $V(s) = V(x, s)|_{x=s}$, $V(s)$ is strictly increasing and convex in $s$.

(3) The first derivative $V_s(s)$ and $V_x(x, s)$ are bounded by 1 for all $(x, s) \in E$.

Proof. To save on notation the optimal stopping boundary $g^*(s)$ will be denoted by $g(s)$. The first derivative of $V(x, s)$ with respect to $s$ is given by

$$V_x(x, s) = \frac{c}{\mu} \left(1 - e^{\frac{2\mu}{2\sigma^2}(g(s)-x)}\right).$$

By smooth pasting, $\lim_{x \to g(s)} V_x(x, s) = 0$, while $\lim_{x \to s} V_x(x, s) = 1$. If $\mu < 0$, $c/\mu < 0$, while $1 - \exp(2\mu(g(s)-x)/\sigma^2) < 0$ and thus $V_x(x, s) > 0$. If $\mu > 0$ the signs switch and $V_x(x, s) > 0$. The second derivative with respect to $x$ is given by

$$V_{xx}(x, s) = \frac{2c}{\sigma^2} e^{\frac{2\mu}{2\sigma^2}(g(s)-x)} > 0.$$ 

Thus $V(x, s)$ is convex in $x$. This proves the first point.

With respect to point two, using the definition of $\alpha$ shows that the value is given by

$$V(s) = V(x, s)|_{x=s} = s + \frac{\sigma^2}{2\mu^2} \left(e^{\frac{2\mu}{2\sigma^2}(-\alpha)} - 1 + \frac{2\mu}{\sigma^2} \alpha\right)$$

$$= s - \frac{\sigma^2}{2\mu} \left(1 + \frac{c}{\mu} \ln \left(1 - \frac{\mu}{c}\right)\right).$$

Since the second term is independent of $s$, $V_s(s) = 1$. The second derivative is $V_{ss}(s) = 0$, proving point two. Point three directly follows from these arguments. \qed

5 The Decision Maker’s Problem

In this Section I derive the solution for the decision maker’s problem. The first Subsection states the problem and provides a first discussion of how the
solution depends on the parameters of the problem. I then analyze a pathological case, which is of particular economic interest. The second Subsection derives the optimal stopping boundary and the value from the problem. I restrict the exposition to the case of a positive drift, $\mu > 0$, which allows me to write the stopping problem as a standard maximization problem. The Main Theorem of this Section is, however, valid for all values of the drift.

5.1 Formulation of the Problem

In this Section I use the sequential structure of the model and nature’s solution to rewrite the decision maker’s problem. Using nature’s value $V(x, s)$ allows me to rewrite the original problem in the following way$^5$:

$$W^*(x, s) = \sup_{\tau} E_{x,s} e^{-\kappa \tau} [X_\tau - \kappa V(X_\tau, S_\tau)]$$  \hspace{1cm} (6)$$

This problem is two-dimensional. It depends both on the initial value of the Brownian motion $x$ and the initial value of the running maximum $s$. This is due to evil self’s objective to maximize the running maximum net of cost.

An important first observation, which has an interesting economic interpretation concerns the weighting factor $\kappa$. From the solution to the subproblem I know that $V(x, s) \geq s$ for all $(x, s) \in E$. By just looking at the decision maker’s value it becomes clear that when $\kappa$ is close to one, regret is overwhelming. As $\kappa$ measures the intensity of regret, I say that the decision maker is overly receptive to regret if $\kappa$ is large. The decision maker, however, is not completely defenseless against her extreme propensity to regret. Even when I take $\kappa$ to one, there is one strategy which guarantees herself a positive utility. It is to preempt the evil self from ever taking control. The only way to do so is to delay the decision indefinitely. This result was obtained by O’Donoghue and Rabin (2001) in a model of procrastination with hyperbolic

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$^5$Recall $V(X_\tau, S_\tau) \equiv \sup_{T} E_{X_\tau,S_\tau} (S_T - cT)$ and $(X_t, S_t)$ is the Brownian motion and its associated maximum process started at $(X_\tau, S_\tau)$. 

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discounting. There a decision maker delays her decision in a stopping game, because she believes she is going to make it tomorrow, which then never happens. I find in my context that regret considerations can lead to the same result.

A solution to the decision maker’s problem consists of a value function $W_s(x, s)$, a continuation and a stopping region. I seek a stopping boundary $s \mapsto h(s)$, which separates these two regions. In principle the continuation region could have one of two shapes: either one stops the first time that $X_t$ falls to $h(S_t)$ or the first time $X_t$ increases to $h(S_t)$. To see that the former is not possible, it suffices to note that the stopping region would be given by $\{ (x, s) : X_t \leq h(S_t) \}$. But then one may pick $x$ such that the payoff upon stopping is negative, $x - \kappa V(x, s) < 0$. This cannot be optimal, because by continuing forever, the decision maker can guarantee herself a non-negative payoff. Therefore the stopping rule resembles the stopping rule from the model without regret. The decision maker stops the first time the value of $X_t$ exceeds $h(S_t)$.

5.2 The Solution to the Decision Maker’s Problem

Based on the argument from the last Subsection, I hypothesize that the decision maker stops the first time the Brownian motion (weakly) exceeds a certain cutoff. The key observation is that the decision maker’s payoff upon stopping depends crucially on whether the evil self continues to increase regret or stops. Intuitively, if the decision maker stops at a value $x$, which is far away from the current value of the maximum process, then the evil self should also stop immediately. The evil self’s decision depends on the gap $s - x$. If the gap is large, it takes a long time until regret increases again and this implies that whatever amount of regret is attained in the future, it is subject to large costs. This suggests that the initial value $s$ of the maximum process, the initial reference point, plays an important role. In particular I hypothesize that if $s$ is large, the decision maker will stop at a value at
which the evil self stops immediately. And when $s$ is small, she will stop at a value, where the evil self feels compelled to increase regret. In the following I assume this is the case and solve the problem under this hypothesis.

To make the derivation of the solution as transparent as possible, I assume for expositional purposes that the drift $\mu > 0$. The Main Theorem in this Section does not depend on the sign of the drift. If, however the drift is positive, I can appeal to a well known result which concerns the Laplace transform of a first passage time. In particular, for $\theta > 0$ and a first passage time $\rho$ to a value $z$, i.e. $\rho = \inf \{t > 0 : X_t = z\}$, I have $E \exp \{-\theta \rho\} = \exp \left\{ - \left( -\mu + \sqrt{\mu^2 + 2\theta \sigma^2} \right) z / \sigma^2 \right\}$. I make use of the Laplace transform to rewrite the decision maker’s optimal stopping problem as a simple maximization problem, as will become clear in the next paragraph. In the sequel I adopt the following convention: if the evil self stops immediately I refer to it as passive, while when it does not I refer to it as active.

The High Region. Let $(x, s)$ be the starting value of the processes. Assume that the decision maker optimally stops at a value $h^*$ at which the evil self stops immediately. This is the case when $h^* \leq g_*(s)$ and the evil self’s value is $V(h^*, s) = s$. Provided that $h^* \leq g_*(s)$, the decision maker’s problem is to maximize $E_{x,s}e^{-\tau \kappa} (X_\tau - \kappa s)$. Since the optimal stopping rule is of the cutoff type, let $h$ be some cutoff. The associated stopping time is $\tau = \inf \{t > 0 : X_t \geq h\}$. Thus the decision maker’s value from stopping at $h$ is $(h - \kappa s)E_{x,s}e^{-\tau \kappa}$. Making use of the Laplace transform, the original stopping problem becomes a simple maximization problem over the cutoff $h$,

$$\max_h e^{b_1(x-h)} (h - \kappa s), \quad (7)$$

---

*see for example Karlin and Taylor (1974) page 362*
subject to $h \leq g_*(s)$ and $b_1$ is given by $b_1 \equiv (-\mu + \sqrt{\mu^2 + 2r\sigma^2})/\sigma^2$. The following Lemma solves for the optimal cutoff $h^*$. The strategy is to first ignore the constraint and to show that the resulting solution does satisfy it. This is true for large values of $s$.

Lemma 5. Let $\mu \geq 0$ and $(x, s)$ be the initial values of the Markov process $(X_t, S_t)$. Let $\bar{s}$ solve $g_*(\bar{s}) = \kappa \bar{s} + 1/b_1$. $\bar{s}$ exists and is unique. Moreover for all $s \geq \bar{s}$ it is optimal to stop the first time $X_t$ reaches the cutoff $h(s) = \kappa s + 1/b_1$, while the optimal value is

$$W(x, s) = \begin{cases} \frac{1}{b_1} e^{b_1(x-(\kappa s+1/b_1))}, & \text{if } x < h(s); \\ x - \kappa s, & \text{otherwise.} \end{cases}$$

Proof. Fix initial values $(x, s)$ and consider the unrestricted problem in (7). The first order condition is

$$-b_1 e^{b_1(x-h)} \left( h - \kappa s - \frac{1}{b_1} \right) = 0.$$

This implies that the optimal cutoff satisfies $h = \kappa s + 1/b_1$. To verify that $h$ is a maximum, take a second order condition and note that

$$b_1^2 e^{b_1(x-h)} \left( h - \kappa s - \frac{1}{b_1} \right) - b_1 e^{b_1(x-h)}$$

is strictly negative at $h$. Thus for a given $s$, $h$ is a valid maximizer. $g'_*(s) = 1$ and $1/b_1 + \kappa s$ has slope $\kappa < 1$. At $\bar{s}$, $g_*(\bar{s}) = 1/b_1 + \kappa \bar{s}$. Hence for $s > \bar{s}$, $h < g_*(s)$ and $h$ also solves the constrained problem. Varying $s$, this defines a function $h(s) = \kappa s + 1/b_1$ and each $h(s)$ is maximizer for $s \geq \bar{s}$. It remains to show that $\bar{s}$ exists and is unique. Existence follows firstly from the fact that $\lim_{s \to 0} g_*(s) = -\alpha < 0$, while $h(0) = 1/b_1$ and secondly from $g'_*(s) = 1$ and $h'(s) = \kappa < 1$ for $\kappa < 1$. Uniqueness follows from the slopes of the two functions. Finally, the optimal value follows from substituting for $h$ into the
objective function.

Remark 6. The decision maker’s problem in the high region is identical to the problem when there is no regret. Introduce the measure $P_y$ where $y = x - \kappa s$. Under measure $y$, with associated expectation operator $E_y$, the objective can be written as $\sup_{\tau} E_y e^{-\tau} X_\tau$, where $X_\tau$ is a Brownian motion, which starts at $y$.

This Lemma determines part of the continuation set, because it applies only in the region where the evil self stops immediately ($s \geq \tilde{s}$). Figure 2 illustrates the continuation and stopping regions. In the high region it is optimal to continue iff $X_\tau < \kappa s + 1/b_1$. Compared to the model without
regret the decision maker continues for longer, because the cutoff is larger than $1/b_1$, that is she behaves more aggressively. The driving force is her regret considerations. In the model without regret stopping at $x$ gives an (undiscounted) value of $x$, but with regret stopping at the same value gives a utility of $x - \kappa s$. It is worthwhile to continue for longer, in order to reduce regret. The class of problems in this region is quite peculiar. The decision maker is convinced, right from the beginning, that the ideal outcome $s$ is high. While this motivates her to continue for longer, her impatience prevents her from attaining the target she sets herself. There is little to gain in this region and the value, for a given initial value $x$, is lowest.

The fact that the evil self incurs sampling costs does not matter in this region, because it stops immediately when the decision maker does so. However, the value $\bar{s}$, which characterizes the region does depend on costs $c$. In the Section on comparative statics, I will detail how $c$ and $r$ qualitatively affect the solution.

**The Intermediate Region.** Let the current best value $s$ satisfy $s < \bar{s}$. If the evil self were to stop, the decision maker would stop as in the high region, along the line $h(s) = 1/b_1 + \kappa s$. This line, however, now lies in the interior of the evil self’s continuation region. Thus the value upon stopping is now given by $h(s) - \kappa V(h(s), s)$ and the evil self attempts to improve upon the best current value. The next Lemma shows that for every fixed $s < \bar{s}$, there exists a unique value $\tilde{h}$ such that it is optimal for the decision maker to stop when $X_t \geq \tilde{h}$ for the first time. Moreover, since this can be done for every $s$, this defines the stopping boundary $h(s)$ for $s < \bar{s}$. Clearly $h(s) < g_*(s)$ for all $s < \bar{s}$. But the Lemma also shows that there exists a point $s^* > 0$ such that $h(s^*) = s^*$. Consequently for $s < s^*$ it is optimal to stop at the diagonal. Figure 3 illustrates the stopping rule in the intermediate region.

**Lemma 7.** Let $\mu \geq 0$, $s < \bar{s}$. The optimal stopping boundary $h(s)$ has the following properties:
Figure 3: The function $h(s)$ for $s \in (s^*, \bar{s})$. In the blue-grey area both the decision maker and the evil self continue.
(1) For each given $s$, there exists a unique value $h(s)$ such that stopping is optimal when $X_t \geq h(s)$ for the first time.

(2) For each given $s$, the optimal cutoff value $h(s)$ is implicitly defined by $h(s) = \kappa V(h(s), s) + \frac{1}{b_1} - \frac{1}{b_1} \lambda V_x(h(s), s)$.

(3) There exists a value $s^*$ such that $h(s^*) = s^*$.

(4) $h(s)$ is strictly increasing in $s$.

For any $s \in (s^*, \tilde{s})$ the optimal stopping time is $\tau = \inf\{t > 0 : X_t \geq h(s)\}$ and the optimal value is given by

$$W(x, s) = \begin{cases} e^{b_1(x-h(s))} (h(s) - \kappa V(h(s), s)), & \text{for } x < h(s); \\ x - \kappa V(x, s), & \text{otherwise.} \end{cases}$$

Remark 8. The proof of (4) is contained in the Appendix, Section 9.2. The rest of the proof is relegated to the proof of Theorem 12. For a given $s$, the claims of the Lemma follow from the solution to the maximization problem

$$\max_{\tilde{h}} e^{b_1(x-\tilde{h})} \left( \tilde{h} - \kappa V(\tilde{h}, s) \right),$$

where I again made use of the Laplace transform for the stopping time. The first order condition reads

$$-b_1 e^{b_1(x-\tilde{h})} \left( \tilde{h} - \kappa V(\tilde{h}, s) - \frac{1}{b_1} + \kappa \frac{1}{b_1} V_x(\tilde{h}, s) \right) = 0. \quad (8)$$

While this region shares the feature with the high region that the initial reference point, $s$, is never improved upon, there is an important difference. This stems from the evil self’s behavior of not stopping immediately. Since the evil self’s problem starts from the value $(X_\tau, S_\tau)$ at which the decision maker stops, she now has an additional possibility for controlling her evil self. Two effects influence the stopping decision. They become apparent if I
rewrite the first order condition (8) in the following way:

\[-b_1(\bar{h} - \kappa V(\bar{h}, s)) + 1 - \kappa V_x(\bar{h}, s) = 0.\] (9)

This equation can be interpreted as a non-arbitrage equation over the state space \(E\). \(b_1\) is a measure of the loss resulting from the choice of a larger stopping time and can be interpreted as a discount rate. The term \(-b_1(x - \kappa V(x, s))\) thus captures the loss from waiting due to discounting. The term \(1 - \kappa V_x(x, s)\) is the rate at which the payoff upon stopping increases. The first derivative \(V_x(x, s)\) embodies the decision maker’s cost from choosing a larger cutoff. They arise because it puts the evil self in a better position, as the gap to the line \(x = s\), at which the evil self can improve, decreases. As the decision maker increases \(x\), her payoff from stopping \(x - \kappa V(x, s)\) increases and thus discounting becomes more of a concern. This discourages continuation. On the other hand, the gains from continuation decrease, because \(V(x, s)\) is convex in \(x\) and so this also discourages continuation. Thus, while the returns from investing into a safe asset which bears interest \(b_1\) increase, the capital gains from holding on to the game decrease and so there is a point \(\tilde{h}\) at which it is optimal to stop.

**Self Control.** To see that the decision maker exerts self control, note that if \(V_x(h, s)\) were zero, the decision maker would continue until a cutoff \(h\) such that \(h - \kappa V(h, s) = 1/b_1\). Costs, \(V_x(h, s)\), are always positive, and so \(\bar{h} < h\). The decision maker stops earlier to keep the gap between \(h\) and \(s\) large, because this hurts the evil self. Thus the decision maker deliberately forgoes a higher payoff from stopping in exchange for lower regret. By stopping at a lower cutoff she is able to discipline her evil self.

**Active versus Passive Evil Self.** To appreciate the effect the evil self has on the decision maker’s behavior it is useful to compare the stopping rule when the evil self is active and does not stop immediately, to the one
where the evil self is passive and does stop. When the evil self is passive, the decision maker stops along the line $\kappa s + 1/b_1$, just as in the high region. Figure 3 shows a case where $h(s) < \kappa s + 1/b_1$. But this is not the only case. The boundary $h(s)$ may also cross this line or lie to the right of it. As $h(s)$ is implicitly given, one can only hope for a sufficient condition for it to be to the left of $\kappa s + 1/b_1$. To this end, let $\mu > 0$ and $x^+$ be the larger solution to the equation $\frac{\mu + \psi}{\mu + \theta} (x + \ln (1 - x)) = 0$. Let $x^-$ be the smaller solution of this equation when $\mu < 0$. The next Lemma provides a sufficient condition for when $h(s)$ lies below $\kappa s + 1/b_1$.

**Lemma 9.** For $\mu > 0$ and $\mu/c \in (0, x^+]$ or $\mu < 0$ and $\mu/c \in [x^-, 0)$, the optimal stopping boundary satisfies $h(s) < 1/b_1 + \kappa s$ for $s \in [s^*, \bar{s}]$.

*Proof.* See Appendix, Section 9.3. \hfill $\square$

The Lemma sheds light on how the attempt of self control and the need to guarantee a positive payoff trade off. If the search cost is close enough to the drift $\mu$, the evil self finds it profitable to continue for a long timespan. This implies that the decision maker is confronted with a large expected regret. As the decision maker can ensure herself a non-negative payoff by never making a decision, it is optimal in this case to continue for long. This pushes the stopping point $s^*$ in the low region to the right of the line $\kappa s + 1/b_1$. Thus when the evil self’s costs are low, the decision maker stops later compared to a situation where the evil self is passive. For moderate values of $\mu/c$, self control implies that the decision maker stops earlier when the evil self is active. Figure 4 illustrates the optimal stopping boundary.

**The Low Region.** It remains to find the point $s^*$. The next Lemma confirms that this point exists and is unique.

**Lemma 10.** Let $\mu \geq 0$ and the initial values $(x, s)$ of the process $(X_t, S_t)$ satisfy $x \leq s < s^*$. There exists a unique value $s^*$ such that stopping
Figure 4: The function $h(s)$ for $\mu/c$ large and $s \in (s^*, \tilde{s})$. 
is optimal the first time $X_t$ hits $s^*$. The optimal stopping time is $\tau = \inf \{ t > 0 : X_t \geq s^* \}$ and the value is given by

$$W(x) = e^{b_1(x-s^*)} (s^* - \kappa V(s^*, s^*))$$

for $x < s^*$.

Remark 11. The proof of this claim is relegated to the proof of the Main Theorem to follow, Theorem 12. Just note that in the current setting, the optimal stopping problem can be written as

$$\max_{s^*} e^{b_1(x-s^*)} (s^* - \kappa V(s^*)),$$

where $V(s^*) \equiv V(s^*, s^*)$. The problem’s first order condition is given by

$$-b_1 e^{b_1(x-s^*)} \left( s^* - \kappa V(s^*) - \frac{1}{b_1} + \kappa \frac{1}{b_1} V_s(s^*) \right) = 0 \quad (10)$$

which implicitly defines the cutoff $s^*$ at which it is optimal to stop. The value follows from substituting the solution into the objective function.

The distinguishing feature of the low region is that the decision maker actually improves over the value of the maximum she started with. She does so despite the fact that she thereby also improves the prospects for her evil self and despite the fact that the evil self’s value improves most at $x = s$. In the low region the gains from continuation outweigh the loss from discounting and the cost from improving the evil self’s outcome.

At this point, it is useful to summarize my results for the optimal stopping boundary $h(s)$. From Lemma 5, 7 and 10. The optimal stopping boundary
is given by

\[
h(s) = \begin{cases} 
-\infty, & \text{for } s < s^*; \\
s^*, & \text{for } s = s^*; \\
\tilde{h}(s), & \text{for } s \in (s^*, \tilde{s}); \\
\frac{1}{b_1} + \kappa s, & \text{for } s \geq \tilde{s}, 
\end{cases}
\]  

\tag{11}

where \(\tilde{h}(s)\) is the boundary defined in Lemma 7.

I next state the Main Theorem.

**Theorem 12.** Let \(c > \mu\). Consider the optimal stopping problem

\[
W_*(x, s) = \sup_{\tau} E_{x,s} e^{-\tau \gamma} (X_\tau - \kappa V(X_\tau, S_\tau)).
\]

Let \(h(s)\) be as in (11) and define the stopping time \(\tau_* \equiv \inf \{t > 0 : X_t \geq h(S_t)\}\). Then (i) \(\tau_*\) is optimal and (ii) the value \(W_*(x, s)\) is given by

\[
W_*(x, s) = \begin{cases} 
e^{b_1(x-s^*)} (s^* - \kappa V(s^*)), & \text{for } x < s^*; \\
e^{b_1(x-h(s))} (h(s) - \kappa V(h(s), s)), & \text{for } x < h(s), s \in (s^*, \tilde{s}); \\
\frac{1}{b_0} e^{b_1(x-\kappa s-1/b_1)}, & \text{for } x < h(s), s > \tilde{s}; \\
x - \kappa V(x, s), & \text{otherwise.}
\end{cases}
\]

If \(c \leq \mu\) the value is infinite.

**Proof.** See Appendix, Section 9.4. \(\square\)

Figure 5 shows the optimal boundary for moderate ratios of \(\mu/c\) (see Lemma 9).

Below \(s^*\) it is optimal to continue. In the middle region, where the evil self is active, the optimally stopped Brownian motion stays short of the diagonal. Once \(s\) is large enough such that the evil self stops immediately, the decision maker continues until \(X_t\) hits the line \(\kappa s + 1/b_1\). While it is clear that the
Figure 5: The optimal stopping boundary for moderate $\mu/c$. 
optimal cutoff with regret exceeds the optimal cutoff without regret in the high region, this is true in general, as the next Corollary shows.

**Corollary 13.** The optimal cutoff with regret exceeds the optimal cutoff without regret for all \((x, s) \in E\).

**Proof.** Since \(\tilde{h}(s) \geq s^*\), it suffices to show that \(s^* > 1/b_1\), because \(1/b_1\) was the cutoff from the model without regret.

Define \(f(s) \equiv s - \kappa V(s) + 1/b_1 V_s(s)\). Observe that the first order condition (10) can be written as \(f(s^*) = 1/b_1\). Using that \(V_s(s) = 1\), this is equivalent to

\[
 s^* = \frac{1}{b_1} - \frac{\kappa \sigma^2}{1 - \kappa 2\mu} \left( 1 + \frac{c}{\mu} \ln \left( 1 - \frac{\mu}{c} \right) \right)
\]

Define \(x \equiv \mu/c\) and \(f(x) \equiv 1 + 1/x \ln(1-x)\). Note that \(f(\mu/c) = 1 + c/\mu \ln(1-\mu/c)\). The function \(f(x)\) is strictly decreasing in \(x\), has value zero at \(x = 0\) and its limit as \(x\) approaches one is negative infinity. Therefore it is positive for \(x \in (-\infty, 0)\) and negative for \(x \in (0, 1)\). If \(\mu/c \in (-\infty, 0)\), it follows that \(\mu < 0\) and thus \(s^* > 1/b_1\). On the other hand, if \(\mu/c \in (0, 1)\), \(1 + \frac{c}{\mu} \ln(1-\frac{\mu}{c})\) is negative and so \(s^* > 1/b_1\). For \(\mu = 0\), a Taylor series expansion of the logarithm around \(\mu/c = 0\) shows that \(\ln(1-\mu/c) \approx -\frac{\mu}{c} - \frac{\mu^2}{2c^2} - \frac{\mu^3}{3c^3} - \ldots\). Thus

\[
 \lim_{\mu \to 0} \frac{\sigma^2}{2\mu} \left( 1 + \frac{c}{\mu} \ln \left( 1 - \frac{\mu}{c} \right) \right) = -\frac{\sigma^2}{4c},
\]

and the claim is proven. \(\square\)

This concludes the characterization of the solution. In the next Section I address the question of the comparative statics of the model.

### 6 Comparative Statics

In this Section I analyze how the solution of the model depends on the underlying parameters of the problem. Parameters of interest are the discount
rate \( r \), search cost \( c \) and the drift \( \mu \). Another parameter is \( \kappa \), which measures the intensity with which regret is felt in relation to the usual payoff component. Although intuitively an increase in search costs \( c \) seems to be equivalent to a decrease in \( \kappa \), it is easy to see that this is not entirely true. In particular, when \( c \) is infinite, it is best for the evil self to stop immediately for all \((x, s) \in E\). This leaves the regret component \( \kappa s \) intact and one cannot expect this to coincide with the solution of the model without regret. If, by contrast, \( \kappa \) is set equal to zero, then clearly the model becomes the model without regret.

In Subsection 6.1 I analyze how \( r \) and \( \kappa \) affect the solution. In Subsection 6.2 search costs are addressed. The final Subsection is concerned with the drift.

### 6.1 Intensity of Regret and the Decision Maker’s Discount Rate

This Section discusses the comparative statics resulting from a change in \( r \) and \( \kappa \). Both parameters do not affect the evil self’s optimal decision: The evil self moves second and so the discounting of the decision maker has no importance for the evil self and neither has the intensity of regret \( \kappa \).

The first Lemma, derives how the decision maker’s strategy changes with the value of the discount rate \( r \).

**Lemma 14.** If \( r \) increases, the decision maker stops earlier for all \((x, s)\) in the continuation region. The decision maker’s value decreases in \( r \).

**Proof.** The derivative of \( \kappa s + 1/b_1 \) with respect to \( r \) is given by \(-1/(b_1^2 \psi)\). Thus the line \( \kappa s + 1/b_1 \) shifts down as \( r \) increases. Consider next the point \( s^* \). Total differentiation of the first order condition immediately shows that

\[
\frac{ds^*}{dr} = -\frac{(1 - \kappa) \frac{1}{b_1 \psi}}{1 - \kappa V_s(s) + \frac{1}{b_1} \kappa V_{ss}(s)}
\]
Corollary 4 showed that $V_s(s) = 1$ and $V_{ss}(s) = 0$. Since $\kappa \in (0, 1)$ this guarantees that the denominator of this fraction is positive and so $ds^*/dr < 0$.

Totally differentiating the equation defining $\tilde{h}$, equation (8), leads to

$$\frac{d\tilde{h}}{dr} = -\frac{(1 - \kappa V_x(\tilde{h}, s))\frac{1}{b_2\psi}}{1 - \kappa V_x(\tilde{h}, s) + \frac{1}{b_1}\kappa V_{xx}(\tilde{h}, s)}.$$  

Corollary 4 showed that $V_x(x, s) \leq 1$ and $V_{xx}(x, s) > 0$ for all $(x, s)$ in the evil self’s continuation region. Together with $\kappa \in (0, 1)$, this implies that the numerator is positive. The same argument shows that the denominator is positive. Therefore $\frac{d\tilde{h}}{dr} < 0$. Thus both derivatives are strictly negative. Finally, the Envelope Theorem shows that the value is decreasing in $r$. \hfill \Box

Thus, the introduction of regret into the stopping problem does not change the comparative statics with respect to the decision maker’s impatience. Just as in the model without regret, the decision maker stops earlier as she becomes more impatient. Note, however, that her cutoff is bounded from below, because any optimal policy always generates non-negative payoffs.

A second parameter of interest, which is unrelated to nature’s stopping decision is $\kappa$, which measures the severity of regret. From the results so far, I know that the introduction of regret induces the decision maker to continue longer. It then stands to reason that an increase in $\kappa$ aggravates the decision maker’s reluctance to abandon the game. The next Lemma shows that this is indeed the case.

**Lemma 15.** An increase in $\kappa$ lowers the value of the decision maker. Moreover, stopping occurs later.

**Proof.** It is clear that the value decreases as $\kappa$ increases. To see that stopping occurs earlier, note that $d/d\kappa(k s + 1/b_1) > 0$. Total differentiation of equation
(17) leads to
\[
\frac{ds^*}{dk} = \frac{V(s^*) - \frac{1}{b_1}V_s(s^*)}{1 - \kappa V_s(s^*) + \frac{1}{b_1}\kappa V_{ss}(s^*)}.
\]
The denominator is again positive by the same argument as in the proof of the previous Lemma. To see that the numerator is positive, note that the proof of Corollary 13 implies \( V(s^*) > s^* > 1/b_1 \) and that \( V_s(s^*) = 1 \). Thus \( \frac{ds^*}{dk} > 0 \). For \( s \in (s^*, \bar{s}) \), the total derivative can be written as
\[
\frac{dh(s)}{dk} = \frac{V(h(s), s) - \frac{1}{b_1}V_x(h(s), s)}{1 - \kappa V_x(h(s), s) + \frac{1}{b_1}\kappa V_{xx}(h(s), s)}.
\]
In the proof of the previous Lemma the denominator was shown to be positive. By value matching \( V(g(s), s) = s \) and \( V(x, s) \) is strictly increasing in \( x \). Therefore \( V(h(s), s) > s > s^* \). By Corollary 4 \( V_x(h(s), s) \leq 1 \), hence the numerator is positive.

\( \square \)

6.2 Search Costs

6.2.1 Evil Self’s Response to a Change in Costs

How does the decision maker’s solution change in the evil self’s search cost? Clearly, if the evil self incurs more cost, then it should stop earlier. But stopping earlier, also affects the decision maker’s behavior. A change in \( c \) affects both the time the decision maker has to continue in order to make up for her regret and her incentives to control her evil self.

I start with the evil self. The following Lemma confirms the intuition that the evil self stops earlier when costs are larger.

**Lemma 16.** The boundary \( g_*(s) \) from the evil self’s solution to its optimal stopping problem is strictly increasing in \( c \). Moreover, the value \( V(x, s) \) is decreasing in \( c \).

**Proof.** Let \( \tilde{c} < c \) and \( V^i(x, s) \) be the value when the search cost is \( i = c, \tilde{c} \). Since \( \tilde{c} < c \), for every stopping time \( \tau > 0 \), \( S_{\tau} - \tilde{c}\tau > S_{\tau} - c\tau \). Let \( \tau_i \) be the
optimal stopping time for costs $i$. I have $V^\tilde{c}(x, s) = S_{\tau_c} - \tilde{c}\tau_c \geq S_{\tau_c} - \tau_c > S_{\tau_c} - c\tau_c = V^c(x, s)$, where the first inequality follows from optimality and the second from $\tilde{c} < c$. This shows that $V^\tilde{c}(x, s) > V^c(x, s)$ and proves that the value is strictly decreasing in search cost $c$ inside the continuation region. The value is constant in costs in the stopping region.

Differentiating the optimal boundary $g_*(s)$ with respect to $c$ shows that

$$\frac{dg_*(s)}{dc} = \frac{\sigma^2}{2c^2} \frac{1}{1 - \frac{\mu}{c}}.$$ 

This expression is strictly positive for all $\mu < c$, which proves the claim. \(\Box\)

The Lemma establishes that as search costs $c$ increase, stopping occurs earlier. This is intuitive, larger costs take away more from any potential future gains, while it does not affect the law of the process $(X_t, S_t)$. To counter this, the evil self reduces the duration of continuation by shifting the boundary $g_*(s)$ up.

### 6.2.2 The Decision Maker’s Response to a Change in Costs

In this Subsection I analyze the decision maker’s response to a change in the evil self’s costs. I first discuss the case of extreme costs. I then proceed to a more general result. The matter is complicated by the fact that the stopping boundary $h(s)$ is only given implicitly.

We have already seen that if the evil self’s costs are close to the drift, the decision maker puts off the decision forever. The other extreme is that the evil self incurs large costs, that is always stops at the value the decision maker stopped. In this case the decision maker’s payoff from stopping at a cutoff $x$ is $x - \kappa s$, for a given initial value $s$ of the maximum process. It is the easy to see that the following policy is optimal: for $s < s^* = 1/b_1$ stop the first time $s^*$ is reached. For $s \in (s^*, s/(1 - \kappa)b_1)$ stop the first time the diagonal $x = s$ is reached, that is at $s$. And for $s > s/(1 - \kappa)b_1$ stop the first time the Brownian motion attains the value $\kappa s + 1/b_1$. Figure 6
Figure 6: The stopping boundary (in blue) with infinite costs $c$. 

$$x = g(s) = s$$

$$x = \kappa s + 1/b_1$$
illustrates the optimal stopping boundary for this case. Note that the evil self’s boundary coincides with the line $x = s$. As for the decision maker, in the low region her boundary very much resembles the one from the model without regret. Observe in particular that there the decision maker stops at the same cutoff as in the model without regret, at $s^* = 1/b_1$. At $s^*$ regret is zero, because the evil self stops immediately and so the same cutoff can be chosen. In the middle region increasing the cutoff towards $x = s$ has no costs but increases the payoff upon stopping. Once the line $x = s$ is reached, the situation changes. Improving over the initial value of the maximum, $s$, now also increases the evil self’s payoff. This was not optimal at $s^*$ and therefore cannot be optimal at $s > s^*$. Therefore the decision maker stops at $x = s$ and there is no regret. In the high region the solution is the same as before.

The analysis of the extreme case shows that $s^*$ tends to $1/b_1$ when $c \to \infty$ and thus for large $c$, $s^*$ decreases. The next Lemma confirms that this is true for all values of the evil self’s costs.

**Lemma 17.** The optimal cutoff $s^*$ is decreasing in $c$. For $s \in (s^*, \bar{s})$, $h(s)$ is non-monotone in $c$.

**Proof.** Totally differentiating the equation defining $s^*$ gives

$$
\frac{ds^*}{dc} = \frac{\kappa \frac{d}{dc} V(s^*)}{1 - \kappa V_s(s^*) + 1/b_1 \kappa V_{ss}(s^*)}.
$$

Since from Lemma 16, $\frac{d}{dc} V^*(s) < 0$ in the evil self’s continuation region, and the denominator is strictly positive, this shows that $\frac{ds^*}{dc} < 0$. In the intermediate region,

$$
\frac{dh(s)}{dc} = \frac{\kappa \frac{d}{dc} \left( V(x, s) - \frac{1}{b_1} V_x(x, s) \right)}{1 - \kappa V_x(x, s) + \frac{1}{b_1} \kappa V_{xx}(x, s)}.
$$

The denominator of this expression is always positive. Consider the derivative
in the numerator evaluated at \( x = g(\tilde{s}) \) and \( \tilde{s} \). One has
\[
\frac{dh(s)}{dc} \bigg|_{x=g(\tilde{s}), s=\tilde{s}} = \frac{\kappa}{b_1(c - \mu)} > 0.
\]

Thus, while an increase in \( c \) shifts \( s^* \) down, \( h(\tilde{s}) \) increases. Therefore the decision maker’s response is non-monotone. She increases her cutoff for some and decreases it for other \( s \).

When the evil self’s costs decrease, the decision maker has to increase her cutoff in order to insure herself a non-negative payoff. Remember that the decision maker can always guarantee herself a payoff of zero by continuing forever. As regret looms large, the stopping decision has to be delayed.

While the decision maker’s reaction is unambiguous with respect to the cutoff \( s^* \), this is not true in the middle region. Figure 7 contrasts two solutions of the model: when \( c \) is infinite and when \( \mu/c \) is moderate. When \( c \) is infinite, the cutoff in the low region is given by \( 1/b_1 \), while when \( \mu/c \) is moderate, the cutoff is \( s^* > 1/b_1 \), but to the left of \( \kappa s + 1/b_1 \). Lemma 16 showed that the cutoff in the low region is monotone in the evil self’s costs. The optimal cutoffs above \( s^* \), however do not share this property. When \( \mu/c \) is moderate, the decision maker stops earlier than in the case of an evil self with infinite costs. More generally, in the middle region the stopping boundaries \( h(s) \) for different values of \( c \) will always cross.

The tradeoffs are best illustrated in the extreme case. There are two opposing forces at work: on the one hand increasing the cutoff is costly, because it increases the stopping time and there is discounting. On the other hand the value upon stopping increases. The non-monotonicity arises, because these two effects play out in a different way when the evil self is active and when it is passive (\( c \) is infinite). To compare the solutions, let \( s \in (s^*, \tilde{s}) \). The key term is found in equation (9),
\[
-b_1 [x - \kappa V(x, s)] + 1 - \kappa V_x(x, s).
\]
Figure 7: Non-monotonicity: The solution for infinite $c$ (blue) versus the solution for moderate $\mu/c$ (red).
When the evil self is passive, the value upon stopping is $x - \kappa s$, while when it is active, the value is $x - \kappa V(x, s)$. Clearly, the decision maker has to continue for longer in the latter case, because the value $x$ at which she breaks even is attained later. Moreover, the loss from discounting is always less when the evil self is active and thus this would lead to a higher cutoff. The offsetting force is the marginal revenue from choosing a higher cutoff. When the evil self is passive, a high cutoff has no effect on the evil self’s outcome, while marginal revenue is one. The active evil self, by contrast, benefits from a higher cutoff and this depresses the decision maker’s marginal revenue, $1 - \kappa V_x(x, s)$. Faced with an active evil self, regret looms large once the line $x = s$ is attained. The decision maker avoids this by choosing a lower cutoff which imposes additional costs on her evil self. This and the shorter continuation compensates her from the loss in value.

6.3 Change in the Drift

The drift is another parameter of interest. In contrast to the discount rate $r$, which directly affects the decision maker but not the evil self, and in contrast to the search cost $c$, which affects the decision maker only through the change it produces in the evil self’s solution, the drift $\mu$ has direct consequences for both players. The leading question is whether the non-monotonicity result for search costs carries over to a change in the drift.

For the evil self, an increase in the drift suggests two main responses: First, the evil self should continue for longer. To see this fix a cutoff $g(s)$ for some drift. At this cutoff, the prospect of improving over the current maximum and the loss associated with the necessary costs from search just balance. If the drift increases, the probability of returning to the line $x = s$ increases, while the value upon stopping $s$, does not change. Therefore in face of a higher drift the decision maker is not indifferent at the cutoff $g(s)$ any more. It is optimal to continue for longer, thus the stopping boundary shifts uniformly down. Equally, a higher drift implies that the game is now
more favorable, and thus the value derived from it has to be higher. The next Lemma confirms these intuitive arguments.

**Lemma 18.** The evil self’s value is increasing in the drift $\mu$. The stopping boundary $g_*(s)$ is strictly decreasing in $\mu$.

**Proof.** Let $\mu > \tilde{\mu}$ and $\tau_\mu$ and $\tau_{\tilde{\mu}}$ be the optimal stopping times. By optimality $S^\mu_{\tau_\mu} - c\tau_\mu \geq S^\mu_{\tau_{\tilde{\mu}}} - c\tau_{\tilde{\mu}}$. By definition $S^\mu_t = \max \{ \mu t + \sigma B_t \} \vee s \geq \tilde{S}^\mu_t$ and strictly so if stopping does not happen immediately. Thus $S^\mu_{\tau_\mu} - c\tau_\mu \geq S^\mu_{\tau_{\tilde{\mu}}} - c\tau_{\tilde{\mu}}$, showing that the value is increasing in $\mu$.

Differentiation of the boundary $g_*(s)$ with respect to $\mu$ shows that

$$
\frac{dg_*(s)}{d\mu} = -\frac{\sigma^2}{2\mu^2} \left( \frac{\mu}{c - \mu} + \ln \left( \frac{c - \mu}{c} \right) \right)
\quad = -\frac{\sigma^2}{2\mu^2} \left( \frac{1}{\frac{\mu}{c} - 1} + \ln \left( 1 - \frac{\mu}{c} \right) \right).
$$

Let $x \equiv \mu/c \in (-\infty, 1)$ and $f(x) \equiv \ln(1-x) + \frac{x}{1-x}$. The first derivative with respect to $x$ is given by $f'(x) = \frac{x}{(1-x)^2}$, showing that $f(x)$ has a critical value at $x = 0$. Noting that $f(0) = 0$, $f(x)$ is increasing for $x \in (0,1)$ and decreasing for $x \in (-\infty,0)$ implies that the critical value is a minimum. Thus $f(x) > 0$ for $x \in (-\infty,0) \cup (0,1)$ and therefore $dg_*(s)/d\mu < 0$. Finally, for $\mu = 0$, expand the logarithm into a Taylor series around zero. One has

$$
\frac{dg_*(s)}{d\mu} \bigg|_{\mu=0} = -\frac{\sigma^2}{4c^2} < 0.
$$

$\Box$

For the decision maker the fact that the evil self’s value increases in the drift $\mu$, implies that the value upon stopping, ceteris paribus, is lower when drift is high. The evil self continues for longer. It faces a more favorable game and thus expected regret is larger. On the other hand, a higher drift is also good news for the decision maker. Her discount factor $b_1$ over the state
space decreases, which implies that larger cutoffs can be obtained at no cost. Furthermore, the incentives for choosing a higher cutoff are strengthened by having to make up for a larger future regret. The countervailing force is self control. Choosing a higher cutoff, does increasingly improve the evil self’s value, as the gap \( s - x \) becomes small. The net effect depends, as in case of search costs, on the region the decision maker is in.

In the high region stopping has to occur later when the drift increases, because the boundary there is given by \( 1/b_1 + \kappa s \) and \( 1/b_1 \) increases in \( \mu \). A higher drift corresponds to an outward shift of this line. When the evil self stops immediately, the decision maker profits from the improved odds without facing adverse consequences from her evil self. For the cutoff in the low region, \( s^* \) the next Lemma shows that it increases in the drift. The reason for this is that in the low region the drift does not affect the rate of increase of the evil self’s value, as \( V_s(s) = 1 \). Thus lower costs of continuation and higher regret both incentivize the decision maker to continue for longer. In the intermediate region, the decision maker’s response is once more ambiguous. A larger drift can lead to earlier stopping decisions.

The change in the decision maker’s value also depends on the region. In the high region, where the evil self stops immediately, a higher drift increases the value, as discounting (over the state space) decreases. In the middle and low regions, the initial position \( x \) plays a prominent role. If the decision maker is far away from the stopping boundary \( h(s) \), the decrease in discounting overwhelms the fact that the evil self imposes more regret. For \( x \) close to the stopping boundary, the effect is reversed and the decision maker’s value decreases in the drift. The next Lemma summarizes.

**Lemma 19.** An increase in the drift \( \mu \) is followed by (i) later stopping in the high region, (ii) later stopping in the low region and (iii) an ambiguous response in the intermediate region. In the intermediate region, if

\[
- \mu ( -c \mu + \mu^2 + c \kappa \psi ) + c \kappa ( c - \mu ) \psi \ln \left( 1 - \frac{\mu}{c} \right) < 0
\]

(12)
the change in the stopping boundary is non-monotone. Moreover, for a given \( \mu > 0 \) there always exists \( \hat{c} \in (\mu, \infty) \) such that \( c < \hat{c} \) implies that (12) is satisfied. The value in the high region is strictly increasing in the drift. For the intermediate and low regions, for every given and fixed \( h(s) \), there exists a value \( \hat{x} \) such that the value is increasing for starting values \( x < \hat{x} \) and decreasing otherwise.

Proof. Point (i) was proven in the text. For the low region, differentiation of the equation defining \( s^* \) yields

\[
\frac{ds^*}{d\mu} = \frac{(1 - \kappa) \frac{1}{b_1 \psi} + \kappa \frac{d}{d\mu} V(s)}{1 - \kappa V_s(s) + \frac{1}{b_1} \kappa V_{ss}(s)} > 0,
\]

and claim (ii) follows from Lemma 18. In the intermediate region, total differentiation of the equation defining \( h(s) \) gives

\[
\frac{dh(s)}{d\mu} = \kappa \frac{d}{d\mu} \left( V(x, s) - \frac{1}{b_1} V_x(x, s) \right).
\]

Let \( F(x, s) \) be equal to the numerator. The left derivative at \( x = g(s)_- \) is given by

\[
F(x, s)|_{x=g(s)_-} = -\frac{\mu (-c \mu + \mu^2 + c \kappa \psi) + c \kappa (c - \mu) \psi \ln \left(1 - \frac{\mu}{c}\right)}{b_1 \psi \mu^2 (c - \mu)},
\]

which is the expression in (12). Observe that if this derivative is negative, then in view of \( \frac{ds^*}{d\mu} > 0 \), it has to be that there exists a point \( z \in (s^*, \tilde{s}) \) at which \( dh(z)/d\mu = 0 \). The boundary rotates in this point \( z \). Fix \( \kappa, \sigma^2 \) and \( r \). Let \( \mu > 0 \) and define \( G(c) \equiv F(x, s)|_{x=g(s)_-} \). One finds

\[
G''(c) = -\frac{\kappa (c + \mu)}{c (c - \mu)^3 b_1} < 0,
\]

so \( G(c) \) is strictly concave. Noting that \( \lim_{c \to \infty} G(c) = 1/b_1 \psi \) and \( \lim_{c \to \mu} G(c) = 0 \).
$-\infty$, shows that for a given $\mu > 0$, there exists a unique $\hat{c}$ such that $G(\hat{c}) = 0$, $G(c) < 0$ for $c < \hat{c}$ and $G(c) > 0$ for $c > \hat{c}$. Thus for all $c < \hat{c}$, $\frac{d h(s)}{d\mu} \bigg|_{x = g(s)} < 0$. This proves (iii).

For the value denote the evil self’s value by $V(h(s), s; \mu)$, underlining the dependence of it on the drift. An application of the Envelope Theorem shows that $\frac{d}{d\mu} W(x, s; \mu)$ equals

$$e^{b_1(x-h(s))} \left( \frac{b_1}{\psi} (h(s) - x) (h(s) - \kappa V(h(s), s; \mu)) - \kappa \frac{d}{d\mu} V(h(s), s; \mu) \right).$$

In the high region $V(h(s), s; \mu) = s$, because the evil self stops immediately and so the change in value is positive. In the low and intermediate regions, from Lemma 18, I know that $\frac{d}{d\mu} V(h(s), s; \mu) > 0$. Since the decision maker’s payoff is positive, $h(s) - \kappa V(h(s), s) > 0$. The sign of $\frac{d}{d\mu} W(x, s; \mu)$ does not depend on the exponential term. The term in brackets is monotonically decreasing in $x$ and strictly negative at $x = h(s)$. Thus there exists $\hat{x} < h(s)$ such that $\frac{d}{d\mu} W(x, s; \mu)$ is positive for $x < \hat{x}$ and negative for $x > \hat{x}$. Finally, in the region of $E$ were both the decision maker and the evil self stop immediately, the value is independent of the drift. \qed

The following two Figures show a numerically obtained solution for a given set of parameters. In Figure 8 the decision maker’s stopping boundary shifts monotonically up as the drift increases. Figure 9 displays the non-monotone case. Observe that the key difference between the two cases is the value of the drift. While in the first Figure the drift is $\mu_{low} = 0.1$ and $\mu_{high} = 0.3$, in the second Figure the numbers are $\mu_{low} = 0.8$ and $\mu_{high} = 0.9$. 

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Figure 8: Monotone Change in boundary in $(s, f(s))$-space. Parameter values: $\sigma = 1$, $r = 1$, $\kappa = 0.5$, $c = 1$, $\mu_{low} = 0.1$, $\mu_{high} = 0.3$. MR refers to the boundary in the middle region, HR to boundary in the high region, given by $\kappa s + 1/b_1$. ES refers to the evil self's stopping boundary. High stands for high drift, low for low drift.

The key ratio is once again $\mu/c$. Intuitively, if the drift is small relative to search costs, an increase in the drift does benefit the evil self, but by much less than when search costs are close in value to the drift. This can be seen from the change in $s^*$ in the two Figures. While in Figure 8, $s^*$ shifts up moderately in order to compensate for more regret, the shift in Figure 9 is more dramatic, although the change in the drift is smaller. The effect of the drift on the evil self's value is thus closely linked to the value of the search costs.
Figure 9: Non-Monotone Change in boundary in \((s, f(s))\)-space. Parameter values: \(\sigma = 1\), \(r = 1\), \(\kappa = 0.5\), \(c = 1\), \(\mu_{\text{low}} = 0.8\), \(\mu_{\text{high}} = 0.9\).

In analogy to a change in search costs, a higher drift implies that regret along the line \(x = s\) looms large. Moreover, any increase in the cutoff improves the evil self’s situation by more when drift is high. This effect is large enough to outweigh the lower discounting and the lower value upon stopping. Consequently boundaries do cross for \(\mu/c\) large enough and the non-monotonicity arises once again.
A Model of Criticism

The model can be more generally interpreted as a game against nature. Nature blames the decision maker for not having made the best choice. This blame game naturally lends itself to an application in political economy. Consider a politician (the decision maker) who has to make a decision the outcome of which evolves over time. The politician’s objective is to find an optimal time at which to make the decision. Once she has made her choice, a second player, the opposition (the evil self) is called upon to move. It is the opposition’s aim to criticize the politician’s action and to decrease her public standing. In doing so, the opposition is subject to two constraints: first, the critique has to relate to the politician’s decision, and second the attack is costly. In particular, the opposition is allowed to monitor the outcome of future decisions at a cost, that is the outcome had the politician made the decision at a different time in the future. As preferences are opposed, it is the best such outcome the opposition compares the politician to. The opposition’s strategy is to choose a time of attack which maximizes the net value of the blame and thus the forcefulness of the critique.

Reinterpretation of my results allows me to explain some salient facts of politics. Hot topics, which are prone to extensive criticism, are often deliberately postponed by the politician. In the extreme, they are put off forever. Control by the opposition (or the media) does lead to better outcomes (for the public). The politician continues for longer, in disrespect of her impatience, because of the blame she is confronted with. The opposition’s role is not always beneficial. The politician sometimes settles for a worse outcome, that is the politician’s behavior is non-monotone in the amount of blame, i.e. the opposition search cost. Finally, politicians have a tendency to preoccupy themselves with low-profile decisions. I view those as the decisions where public expectations are low, that is the initial reference point is low. These provide the politician with the largest utility (for a given and fixed initial value $x$ of the Brownian motion), which may help explain why politician’s
prefer them to large reforms.

The literature on blame finds that it can either improve or reduce the quality of a decision (e.g. Baldwin (2001)), a result I obtain in my model of criticism. Blame is an especially prevalent feature in politics. Anand (1998) develops a static model which formulates blame in a game theoretic context. She finds that blame may block rational choice, but also helps in maximizing social welfare by resolving social dilemmas.\(^7\) In a different context, stressing the importance of blame in politics, Gilmour (2003) addresses the question of why Presidential vetoes occur in the US. She shows that the main reason is that Congress passes bills which it is known will be vetoed. The opposition uses those bills to, firstly, tie the president to a policy and, secondly, to blame her for not enacting a popular policy. Finally DeScioli and Bokemper (2014) propose a mechanism designed to avoid excessive criticism of politicians. In addition to these, in an economic context, Selten (2001) examines how the optimal bidding strategy in a first price sealed bid auction changes, if the bidders are subject to blame.

\section{Conclusion}

In this paper I propose a new way to lend a role to regret over the future, which empirical work finds to be important. I introduce a new parameter into the decision maker’s utility function. The time during which she tracks the prices once she has stopped. This time was taken to be a stopping time and the uncertainty about it is revealed only after the decision maker

\footnote{The author discusses a case study, the outcome of a policy of the Thatcher government to reduce waiting lists in the British health system (NHS). This decision was partly made to show that health is an important issue for the conservative party, although waiting lists only showed a slight uptick in the years after the reform was enacted. The issue was dramatized by the opposition and the press. The loss in the conservative’s political standing was considerable and was due to a lack in anticipation of the public’s forceful reaction. my model suggests, that had the government anticipated this, it would have delayed the decision and settled with a better policy.}
has stopped. I captured the decision maker’s preferences by introducing an evil self of the decision maker, which maximizes the decision maker’s regret. One result I obtain is that the decision maker continues for longer. This provides one possible explanation for why retailers offer price guarantees, or why applications which offer price tracking and buying recommendations enjoy such a popularity.

A second result is that the decision maker has ways to influence the amount of regret she is confronted with. She exerts self control. In choosing the value at which to stop, she takes into account that large values benefit the evil self, provided that the evil self does not stop immediately. Thus she adjusts her decision and stops earlier. This is one way of controlling the evil self. When the evil self faces low costs this becomes less of a concern. Instead the decision maker starts to shield herself from regret and to ensure her a non-negative payoff. This is accomplished by later and later stopping decisions. I showed that this can become extreme. When the evil self has extremely low costs, the decision maker never stops, she puts off the decision forever. The literature on procrastination reached a similar result in a different setting. I contribute to this literature by demonstrating that regret is another source which may motivate individuals to postpone decisions forever.

I also investigated how the different parameters of my model affect the solution. Of particular interest are the discount rates of the decision maker and the search cost of the evil self. I showed that higher search costs of the evil self, provoke a non-monotone response by the decision maker. This implies that more regret does not always lead to a more aggressive behavior in the sense of a longer period of continuation. This carries over to a change in the drift. While search costs affect the decision maker only through the evil self’s reaction, the drift does also directly affect her behavior. The prospects from continuation become better for both. However, the decision maker has to take into account that longer continuation does improve the evil self’s situation. Regret will loom large. Therefore it can happen that the decision
maker attempts to keep his regret at bay. This warrants earlier stopping, which introduces the non-monotonicity. Thus both a change in the drift and in the search cost lead to a non-trivial response by the decision maker. As a consequence, any design to limit regret will turn out to be complicated.

Another interesting application I studied was in the field of political economy. My model can be more generally interpreted as a game against nature. I show how the stopping decision of a politician is influenced by the criticism she faces. I adopted the convention of identifying the source of criticism as the opposition. More generally one may think of it as the current environment, including the opposition, the press and the current public opinion. I argue that my model explains two salient features of politics: high stake decisions, which carry the risk of large criticism are put off. And criticism improves the quality of the politician’s performance.

The effect of regret on consumers has important consequences for the other side of the market. Therefore understanding the channel through which regret considerations affect the consumers’ behavior is highly relevant for the design of mitigation strategies by retailers, or political platforms by politicians in office. In this paper I assess some of the main questions: the channels through which regret delays decisions, the situations in which decisions are increasingly delayed, how the optimal stopping time is affected by the discount rates, the value of the reference point and the drift. There are still plenty of avenues for future research. For example what is an optimal price setting strategy if consumers experience regret and what are the inefficiencies that arise.

References


Paul Viefers and Philipp Strack. A bird in the hand is worth two in the bush: On choice behavior in an optimal stopping task. 2014.


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9 Appendix

9.1 Proof of Theorem 3

9.1.1 Verification.

Let \( g(s) \) be any solution to the differential equation in equation (4) which does not cross the diagonal, i.e. \( g(s) < s \) for all \( s \). Let \( V(x, s) \) solve the system in equation (3) given the boundary \( g \). Define the stopping time 
\[
\tau_g = \inf \{ t > 0 : X_t \leq g(S_t) \}.
\]

Consider the process \( (V(X_t, S_t))_{t \geq 0} \). An application of Ito’s formula to this process (see Graversen and Peskir (1998) for details), given a solution \( g(s) \) to (4), shows that 
\[
V(x, s) + \int_0^t \mathbf{1}_{\{X_u \leq S_u\}} \mathbb{L}_X V(X_u, S_u) du + \int_0^t \sigma \mathbf{1}_{\{g(S_u) < X_u \leq S_u\}} \frac{\partial V}{\partial x}(X_u, S_u) dB_u + \int_0^t \frac{\partial V}{\partial s}(X_u, S_u) dS_u 
\]
(13)

The process \( \{S_u\}_{u \geq 0} \) does not increase off the diagonal, while on the diagonal we have the normal reflection principle. Thus the last integral equals zero. The candidate value function satisfies the system in equation (3), hence the integral in the second line is equal to \( c \) for all \( u \). When \( g(S_u) \geq X_u \), the evil self is required to stop, which gives a payoff of \( S_u \), thus \( V(X_u, S_u) = S_u \). Applying the infinitesimal generator to this expression shows that \( \mathbb{L}_X V(X_u, S_u) = 0 \). Define the processes \( M_t \equiv \sigma \int_0^t \mathbf{1}_{\{g(S_u) < X_u \leq S_u\}} \frac{\partial V}{\partial x}(X_u, S_u) dB_u \). Consider
any stopping time $\tau$. Since $V(X_u, S_u) \geq S_u$ for all $u \geq 0$,

$$E_{x,s}(S_\tau - c\tau) \leq E_{x,s}(V(X_\tau, S_\tau) - c\tau) = V(x, s) + E_{x,s} \left( \int_0^\tau 1_{\{X_u > g(S_u)\}} cdu + M_\tau - c\tau \right),$$

Noting that $c > 0$, one finds that for any stopping time $\tau$,

$$E_{x,s}(S_\tau - c\tau) \leq V(x, s) + E_{x,s}M_\tau.$$

The process $M_{\tau \wedge t}$ is a local martingale and as the next Lemma shows a martingale.

**Lemma 20.** The process $M_{\tau \wedge t}$ is a martingale for any time with finite expectation.

**Proof.** The Burkholder-Davis-Gundy inequality states that there exist constants $k_m$ and $K_m$ such that

$$k_mE_{x,s}\langle M \rangle^m_\rho \leq E_{x,s} \left( \sup_{0 \leq u \leq \rho} M_u \right)^{2m} \leq K_mE_{x,s}\langle M \rangle^m_\rho$$

for any $m$ and stopping time $\rho$ and $\langle \cdot \rangle$ denotes the quadratic variation of the Itô integral $M_t$.

I choose $m = 1/2$ and show that $I \equiv E_{x,s} \int_0^\rho 1_{\{X_u > g(S_u)\}} V_x(X_u, S_u)^2du < \infty$, which implies that $E_{x,s}M_\rho^2 = \sigma^2E_{x,s} \int_0^\rho 1_{\{g(s) < X_u\}} V_x(X_u, S_u)^2du$ and $E_{x,s}M_\rho^2 = E_{x,s}\langle M \rangle_\rho$. I have

$$E_{x,s} \left( \sup_{0 \leq u \leq \rho} M_u \right) \leq K_mE_{x,s} \sqrt{\langle M \rangle_\rho} \leq K_m \sqrt{E_{x,s}\langle M \rangle_\rho},$$

by Jensen’s inequality. Since the integral in $I$ is always non-negative, I drop the indicator function. In Corollary 4 I showed that $V(x, s) \leq V_x(x, s)|_{x=s} = 1$ for all $(x, s)$ in the evil self’s continuation region. So I have
\[ I \leq E_{x,s} \int_0^\rho du = E_{x,s} \rho. \]

The optimal time \( \tau_* \) is given by \( \tau_* = \inf \{ t > 0 : S_t - X_t = \alpha \} \). Let \( T_a = \inf \{ t > 0 : S_t - X_t = a \} \). In Chapter three I show

\[
E_{x,s} T_a = \frac{\sigma^2}{2\mu^2} \left( e^{2\mu a} - e^{2\mu (s-x)} \right) + \frac{s - x - a}{\mu} 
\]

for \( x \in (s-a, s) \). As \( c > \mu, \alpha < \infty \). Choose \( a = \alpha + \epsilon \) for some \( \epsilon > 0 \). Then \( E_{x,s} \tau_* < E_{x,s} T_a < \infty \). Picking \( \rho \) from this class of stopping times proves \( I < \infty \). Since \( E(\sup_{0 \leq u \leq \rho} M_u) < \infty \). This shows the claim. \( \square \)

The Optional Stopping Theorem implies that \( EM_0 = EM_\tau \). Clearly \( EM_0 = 0 \) and thus

\[
E_{x,s} V(X_\tau, S_\tau) \leq V(x, s) 
\]

for all stopping times \( \tau \). Thus taking a supremum over stopping times \( \tau \)

\[
\sup_\tau E_{x,s} (S_\tau - c\tau) \leq V(x, s). 
\]

Observe that the value function \( V(x, s) \) depends on the boundary \( g \), which was left unspecified so far. To make the dependence explicit, denote the value by \( V_g(x, s) \). Finally, denote by \( V_*(x, s) = \sup_\tau (S_\tau - c\tau) \) the solution to the optimal stopping problem. It is known from Markov process theory that \( V_*(x, s) \) is the smallest supermartingale dominating the gain function, \( s \), on the state space \( E \). The candidate value function \( V_g(x, s) \) is decreasing in \( g \),

\[
\frac{\partial V_g(x, s)}{\partial g} = \frac{c}{\mu} \left[ e^{2\mu (g(s)-x)} - 1 \right] < 0. 
\]

Therefore, since \( V_g(x, s) \) is decreasing in \( g \), one is forced to pick the largest such boundary, \( g_*(s) \) which does not cross the diagonal. Thus \( \sup_\tau E_{x,s} (S_\tau - c\tau) \leq \inf_g V_g(x, s) = V_{g*}(x, s) \). Finally, the reverse inequality follows by
noting that \( \sup E_{x,s}(S_t - cT) \geq E_{x,s}(S_{\tau_x} - c\tau_g) = E_{x,s}V(X_{\tau_x}, S_{\tau_x}) - c\tau_g \), which equals \( V_{g_x}(x,s) \) by the previous arguments.

### 9.2 Proof of Lemma 7

The proof of all points except point 4 is delegated to the proof of the Main Theorem.

For point four, let \( h \) be a cutoff for a given \( s \). The first order condition can be written as

\[
-b_1 [h - \kappa V(h, s)] + 1 - V_x(h, s) = 0.
\]

Total differentiation of this equation with respect to \( h \) and \( s \) and making use of the identities \( V_s(x, s) = 1 - V_x(x, s) \), \( V_{xs}(x, s) = -V_{xx}(x, s) \) gives

\[
\frac{dh}{dh} = \kappa \frac{1 - V_x(x, s) + \frac{1}{b_1}V_{xx}(x, s)}{1 - \kappa V_x(x, s) + \kappa \frac{1}{b_1}V_{xx}(x, s)}. \tag{14}
\]

The denominator is positive, because Corollary 4 showed that \( V_x(x, s) \leq 1 \) for all \( (x, s) \in E \) and \( V_{xx}(x, s) > 0 \). To save on notation define \( G(x, s) \equiv 1 - V_x(x, s) + 1/b_1 V_{xx}(x, s) \). Using the definition of \( \alpha = -\frac{\sigma^2}{2\mu} \ln(1 - \mu/c) \) and the fact that \( g(s) = s - \alpha \), shows

\[
G(s, s) = \frac{2c}{b_1 \sigma^2} \left( 1 - \frac{\mu}{c} \right) = \frac{2c}{-\mu + \psi} - \frac{2\mu}{-\mu + \psi} > 0
\]

and

\[
G(g(s), g(s)) = 1 + \frac{2c}{-\mu + \psi} > 0.
\]

From these two equations one finds \( G(g(s), g(s)) > G(s, s) \). The first derivative of \( G(x, s) \) with respect to \( x \) is given by

\[
G_x(x, s) = -V_{xx}(x, s) + \frac{1}{b_1} V_{xxx}(x, s).
\]
Since $V_{xxx}(x, s) = -\frac{2\sigma^2}{\sigma^2} V_x(x, s)$, I obtain

$$G_x(x, s) = -V_{xx}(x, s) \frac{\mu + \psi}{-\mu + \psi} < 0,$$

where the sign follows from noting that $\psi \geq |\mu|$. Thus, $G(x, s)$ has a maximum at $x = g(s)$, is strictly decreasing and attains a value $G(s, s) > 0$ at $x = s$. Therefore $G(x, s) > 0$ for all $x \in (g(s), s]$ and all $s$. This establishes that $\frac{dh}{ds} > 0$. Since $s \in (s^*, \tilde{s})$ was arbitrary, the optimal stopping boundary is strictly increasing in $s$ on this interval.

### 9.3 Proof of Lemma 9

In the intermediate region the optimal cutoff $h(s)$ for any given $s$, satisfies $h(s) = \frac{1}{b_1} + \kappa V(h(s), s) - \frac{1}{b_1} \kappa V_x(h(s), s)$. Using the functional forms for the value function and its derivative, one obtains

$$h(s) = \frac{1}{b_1} + \kappa s + \frac{c}{\mu} F(h, s),$$

with $F(x, s) \equiv \frac{\mu + \psi}{2\psi^2} \left( e^{\frac{2\psi}{\sigma^2} (g(s) - x)} - 1 \right) + x - g(s)$. The first and second derivative with respect to $x$ is given by

$$F_x(x, s) = -\frac{\mu + \psi}{-\mu + \psi} \frac{2\psi}{\sigma^2} e^{\frac{2\psi}{\sigma^2} (g_x - x)} + 1$$

$$F_{xx}(x, s) = -\frac{\mu + \psi}{-\mu + \psi} \frac{2\psi}{\sigma^2} e^{\frac{2\psi}{\sigma^2} (g_x - x)}.$$

Since $\psi = \sqrt{\mu^2 + 2r\sigma^2} \geq |\mu|$, the function $F(x, s)$ is concave when $\mu < 0$ and convex when $\mu > 0$. Observe also that $F(g, g) = 0$. To sign $F(x, s)$, I consider the value of $F(x, s)$ at the other endpoint of the interval, where

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x = s. Define $\beta \equiv \frac{\mu + \psi}{-\mu + \psi}$ and $G(x) \equiv \beta x + \ln(1 - x)$. Note that

$$F(x, s)|_{x=s} = -\frac{\sigma^2}{2\mu} G(\mu/c).$$

(1) Case $\mu > 0$: This implies $\beta > 1$. The function $G(x)$ has the derivatives $G'(x) = \beta - 1/(1 - x)$ and $G''(x) = -1/(1 - x)^2$. Thus $G(x)$ is strictly concave and attains a maximum at $\tilde{x} = 1 - 1/\beta > 0$. Since $G(0) = 0$, and $\lim_{x \to 1} G(x) = -\infty$, this implies that there exists $x^+ > \tilde{x}$ such that $G(x^+) = 0$. Therefore $G(x) > 0$ for $x \in (0, x^+)$ and $G(x) < 0$ for $x \in (x^+, 1)$. So,

$$F(x, s)|_{x=s} \begin{cases} > 0 & \text{if } \frac{\mu}{c} \in (x^+, 1) \\ < 0 & \text{if } \frac{\mu}{c} \in (0, x^+) \end{cases}.$$

Thus, if $\mu/c \in (0, x^+)$, $F(s, s) < 0$ and $F(x, s)$ is strictly convex. Then $F(g, g) = 0$ implies that $F(x, s) < 0$ for all $x \in [g(s), s]$. From equation (15) it follows that $h(s) < 1/b_1 + \kappa s$.

(2) Case $\mu < 0$: This implies $\beta < 1$, that the function $G(x)$ is strictly concave and that reaches a maximum at $\tilde{x} = 1 - 1/\beta < 0$. As before $G(0) = 0$. Thus there exists $x^- < 0$ such that $G(x^-) = 0$. Therefore $G(x) < 0$ for $x \in (-\infty, x^-)$ and $G(x) > 0$ for $x \in (x^-, 0)$. It follows that

$$F(x, s)|_{x=s} \begin{cases} < 0 & \text{if } \frac{\mu}{c} \in (-\infty, x^-) \\ > 0 & \text{if } \frac{\mu}{c} \in (x^-, 0) \end{cases}.$$

$F(g, g) = 0$, strict concavity and $F(s, s) > 0$ imply that $h(s) < 1/b_1 + \kappa s$ for $\mu/x \in (x^-, 0)$.

### 9.4 Proof of Theorem 12

The proof proceeds in several steps. In the first Subsection I set up the free boundary problem associated with the stopping problem. I impose value
matching and smooth pasting. The resulting candidate value function and the equations determining the cutoffs are shown to be equal to those in Lemmata 5, 7 and 10. After this I prove that the candidate value function defines unique cutoffs, which maximize the value of the candidate function. In the second Subsection I verify that the candidate value function is indeed the solution to the optimal stopping problem.

9.4.1 The Free Boundary Problem

The decision maker’s value \( W^*(x, s) \) is given by

\[
W^*(x, s) = \sup_{\tau} e^{-r\tau} (X_\tau - \kappa V(X_\tau, S_\tau)).
\]

The strategy of proof is to derive a candidate value function \( W(x, s) \) solving the problem using Markov theory. It is clear that \( W(x, s) \) solves the following system

\[
\begin{align*}
\mathbb{L}_x W(x, s) & = rW(x, s) \quad x \leq h(s) \\
W(h(s), s) & = h(s) - \kappa V(h(s), s) \\
W_x(h(s), s) & = 1 - \kappa V_x(h(s), s),
\end{align*}
\]

where \( s \to h(s) \) is a boundary to be derived.

**Lemma 21.** The solution \( W(x, s) \) to this system coincides with the value function from Theorem 12.

**Proof.** The solution to the second order differential equation is given by \( Z(x, s) = C_1 e^{b_1 x} + C_2 e^{b_2 x} \). \( b_1 \) and \( b_2 \) were defined in the text and are the solutions to the characteristic equation

\[
b^2 + \frac{2\mu}{\sigma^2} b - \frac{2r}{\sigma^2} = 0,
\]

(16)

As \( X_t \) becomes large and negative, the term \( C_2 e^{b_2 x} \) is unbounded, while the
value is bounded, thus \( C_2 = 0 \). The value matching and smooth pasting condition imply

\[
\begin{align*}
h(s) - \kappa V(h(s), s) &= C_1 e^{b_1 h(s)} \\
1 - \kappa V_x(h(s), s) &= b_1 C_1 e^{b_1 h(s)}
\end{align*}
\]

and thus

\[
h(s) - \kappa V(h(s), s) - \frac{1}{b_1} + \frac{1}{b_1} \kappa V_x(h(s), s) = 0
\]

But this equation, is equivalent to the first order conditions from Lemmata 5, 7, 10.

The next Lemma shows that the cutoffs defined in equation (17) exist, are unique and maximize the candidate value function. This step proves the assertions in Lemmata 5, 7 and 10.

**Lemma 22.** (1) Let \( s < s^* \). There exists a unique value \( s^* \), which maximizes \( W(x, s) \).

(2) Let \( s \in (s^*, \bar{s}) \). For each given \( s \), there exists a unique value \( \bar{h} \), which maximizes \( W(x, s) \).

(3) Let \( s \geq \bar{s} \). For each given \( s \), there exists a unique value \( \tilde{h} \), which maximizes \( W(x, s) \).

**Proof.** Define

\[
f(x, s) \equiv x - \kappa V(x, s) + \frac{1}{b_1} \kappa V_x(x, s)
\]

and observe that equation (17) can be written as \( f(h, s) = 1/b_1 \). The function \( f(x, s) \) has the the following properties: First

\[
f_x(x, s) = 1 - \kappa V_x(x, s) + \frac{1}{b_1} \kappa V_{xx}(x, s) \geq 1 - \kappa
\]

for all \( (x, s) \in E \), because Corollary 4 established that \( V_x(x, s) \leq 1 \) and
\(V_{xx}(x, s) > 0\). Second,

\[
f_{xx}(x, s) = -\kappa V_{xx}(x, s) + \frac{1}{b_1} \kappa V_{xxx}(x, s)
\]

\[
= -\kappa V_{xx}(x, s) \frac{\mu + \psi}{-\mu + \psi} < 0.
\]

Thus \(f(x, s)\) is strictly increasing and concave in \(x\). I next address the different regions.

(1) \(s \geq \tilde{s}\): The first order condition is \(h(s) = 1/b_1 + \kappa s\). For \(x < h(s)\), \(x < 1/b_1 + \kappa s\), while for \(x > h(s)\) one obtains the reverse inequality and so a solution exists.

(2) \(s \in (s^*, \tilde{s})\): At \(\tilde{s}\), \(g(\tilde{s}) = h(\tilde{s}) = 1/b_1 + \kappa \tilde{s}\). Moreover, by value matching \(V(g(s), s) = s\). Smooth pasting implies \(V_x(g(s), s) = 0\), showing that

\[f(g(s), s) = g(s) - \kappa s\]

for all \(s\). The first derivative of \(f(g(s), s)\) with respect to \(s\) is given by \(f_s(g(s), s) = 1 - \kappa > 0\). But, since \(f(g(\tilde{s}), \tilde{s}) = 1/b_1\) this implies that for \(s < \tilde{s}\), \(f(g(s), s) < 1/b_1\) and stopping at the evil self’s boundary \(g(s)\) is not optimal in the middle region. The fact that \(f(x, s)\) is strictly increasing and concave implies that there exists a unique cutoff \(h(s)\) for a given \(s\), once it is shown that \(f(s, s) > 1/b_1\). In the next point it is proven that there exists a unique solution \(s^*\) to the equation \(f(s^*) \equiv f(x, s^*)|_{x=s^*} = 1/b_1\). The function \(f(s)\) is strictly increasing in \(s\). Therefore \(f(s, s) > 1/b_1\) for all \(s > s^*\).

(3) \(s = s^*\): Define \(f(s) \equiv f(x, s)|_{x=s}\). The first order condition for \(s^*\) can be written as \(f(s^*) = s - \kappa V(s^*) + \kappa 1/b_1 V_x(s^*)\). Using that \(V_x(s) = 1\) for all \(s\), one obtains

\[f'(s) = 1 - \kappa, \quad f''(s) = 0.\]

\(V(s) \geq s\) for all \(s\) implies \(f(s) \leq (1-\kappa)s + \kappa 1/b_1\). Letting \(s\) drop to zero shows \(f(0) \leq \kappa 1/b_1 < 1/b_1\), because \(\kappa \in (0, 1)\). Then, by the Intermediate Value Theorem and the fact that \(f(s)\) is strictly increasing in \(s\), there exists a unique
solution $s^*$ to the equation $f(s^*) = 1/b_1$. This concludes the proof.

9.4.2 The Verification.

Denote by $W_*(x,s)$ the value function associated to the optimal stopping problem, that is $W_*(x,s) = \sup_T E_{x,s} e^{-rT} (X_T - \kappa V(X_T, S_T))$. I verify in this Subsection that the candidate value function $W(x,s)$ satisfies $W_*(x,s) = W(x,s)$.

An application of Ito’s formula to the process $(e^{rt}W(X_t, S_t))_{t \geq 0}$ leads to the following equation:

$$
e^{-rt}W(X_t, S_t) = W(x, s) + P_t + Q_t + M_t$$

$$+ \int_0^t 1_{\{X_u<h(S_u)\}} e^{-ru} (\mathbb{L}_X W(X_u, S_u) - rW(X_u, S_u)) \, dB_u$$

Here $\mathbb{L}_X = \mu \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2}$ is the generator associated with the Brownian motion $X_t$ and I have made use of the following definitions:

$$P_t \equiv \int_0^t 1_{\{X_u<h(S_u), S_u \in (s^*, \bar{s})\}} e^{-ru} \times$$

$$\times (-rX_u + \mu + \kappa (rV(X_u, S_u) - \mathbb{L}_X V(X_u, S_u))) \, du,$$

$$M_t \equiv \int_0^t 1_{\{X_u<h(S_u)\}} e^{-ru} \frac{\partial W}{\partial x}(X_u, S_u) \sigma dB_u,$$

and

$$Q_t = \int_0^t 1_{\{g(S_u) \geq X_u > h(S_u), S_u \leq \bar{s}\}} e^{-ru} (-rX_u + \mu + \kappa rS_u) \, du$$

$$+ \int_0^t 1_{\{g(S_u) < X_u, S_u \geq \bar{s}\}} e^{-ru} \times$$

$$\times (-rX_u + \mu + \kappa (rV(X_u, S_u) - \mathbb{L}_X V(X_u, S_u))) \, du.$$
and $Q_t$ are non-positive.

**Lemma 23.** The processes $P_t$ and $Q_t$ are non-positive.

**Proof.** A sufficient condition for $P_t$ to be non-positive is that

$$X_u > \frac{\mu}{r} + \kappa \left( V(X_u, S_u) - \frac{1}{r} \mathbb{L}_X V(X_u, S_u) \right)$$

for all $u$ such that $X_u > h(S_u)$ and $S_u \in (s^*, \tilde{s})$. In the intermediate region, \( \mathbb{L}_X V(x, s) = c \). Define \( f(x, s) \equiv x - \kappa V(x, s) - \frac{\mu - \kappa c}{r} \). Using the properties of $V(x, s)$ from Corollary 4, one has $f_x(x, s) = 1 - \kappa V_x(x, s) \geq 1 - \kappa$ and $f_{xx}(x, s) = -\kappa V_{xx}(x, s) < 0$, establishing that $f(x, s)$ is strictly increasing and concave in $x$. Therefore $f(h(s), s) > 0$ implies that $f(x, s) > 0$ for all $(x, s) \in E$. The first order condition for $h(s)$ was given by $h(s) - 1/b_1 - \kappa V(h(s), s) + 1/b_1 \kappa V_x(h(s), x) = 0$. Subtracting this from the desired inequality $f(h(s), s) > 0$ gives

$$\frac{1}{b_1} - \frac{\mu}{r} + \kappa \frac{c}{r} - \frac{1}{b_1} \kappa V_x(h(s), s) > 0$$

(19)

Since $h(s)$ is inside the evil self’s stopping region, $c = \mathbb{L}_X V(h, s) = \mu V_x(h, s) + \frac{1}{2} \sigma^2 V_{xx}(h, s)$. The second derivative of the value with respect to $x$ is non-negative. Therefore $c > \mu V_x(h, x)$. So the inequality in (19) is implied by the inequality

$$\left( \frac{1}{b_1} - \frac{\mu}{r} \right) (1 - \kappa V_x(h(s), s)) > 0.$$ 

From the characteristic equation (16), one finds that

$$b_1^2 + \frac{2\mu}{\sigma^2} b_1 - \frac{2r}{\sigma^2} = 0 \implies \frac{1}{b_1} > \frac{\mu}{r},$$

which concludes the proof that $P_t$ is non-positive. 

For the process $Q_t$, consider the first integral. A sufficient condition for
this integral to be non-negative is

\[ X_u > \frac{\mu}{r} + \kappa S_u \]

for all \( u \) such that \( g(S_u) > X_u > h(S_u) \) and \( S_u > \tilde{\sigma} \). It is sufficient to show \( h(S_u) > \mu/r + \kappa S_u \). In the high region \( h(S_u) = 1/b_1 + \kappa S_u \). Therefore the inequality is implied by

\[ h(S_u) = \frac{1}{b_1} + \kappa S_u > \frac{\mu}{r} + \kappa S_u, \]

which follows from \( 1/b_1 > \mu/r \).

A sufficient condition for the second integral in \( Q_t \) to be non-negative is given by

\[ X_u > \frac{\mu}{r} + \kappa \left( V(X_u, S_u) - \frac{1}{r} \mathbb{L}_X V(X_u, S_u) \right) \]

for all \( u \) such that \( g(S_u) < X_u \) and \( S_u \geq \tilde{\sigma} \). Define, as before, \( f(x, s) \equiv x - \kappa V(x, s) - \frac{\mu}{r} + \frac{1}{r} \mathbb{L}_X V(x, s) \) and note that the desired inequality can be written as \( f(x, s) > 0 \). By definition \( \mathbb{L}_X V(x, s) = \mu V_x(x, s) + 1/2\sigma^2 V_{xx}(x, s) \). While smooth pasting guarantees that the first derivative of \( V(x, s) \) is smooth at the boundary \( g(s) \), this is in general not true for the second derivative. Indeed \( \lim_{x \uparrow g(s)} V_{xx}(x, s) = 0 \), while \( \lim_{x \downarrow g(s)} V_{xx}(x, s) = \frac{2\kappa}{\sigma^2} > 0 \). From this one obtains \( \lim_{x \uparrow g(s)} f(x, s) > \lim_{x \downarrow g(s)} f(x, s) \). As shown before, \( f(x, s) \) is strictly increasing in \( x \) within the continuation region of the evil self. Combining this with the last inequality implies \( f(x, s) > \lim_{y \uparrow g(s)} f(y, s) \) for all \( x \in (g(s), \tilde{\sigma}] \). For \( y \in [h(s), g(s)] \),

\[ f(y, s) = y - \kappa s - \frac{\mu}{r} > h(s) - \kappa s - \frac{1}{b_1} = 0, \]

because \( h(s) \) satisfies the optimality condition \( h(s) = 1/b_1 + \kappa s \) and \( 1/b_1 > \)
The function \( f(y, s) \) is increasing in \( y \). Therefore
\[
f(x, s) > \lim_{y \downarrow g(s)} f(y, s) > \lim_{y \uparrow g(s)} f(y, s) > f(h(s), s) = 0
\]
for all \( x \in (g(s), s] \) and all \( s \). This proves the claim.

The process \( M_t \) in equation (18) is a local martingale. Choose a localization sequence \((\sigma_n)_{n \geq 1}\). Let \( \tau \) be any stopping time. Observe that \( W(x, s) \geq x - \kappa V(x, s) \). Thus
\[
E_{x,s}e^{-\tau \wedge \sigma_n} \left( X_{\tau \wedge \sigma_n} - \kappa V(X_{\tau \wedge \sigma_n}, S_{\tau \wedge \sigma_n}) \right) \leq E_{x,s}e^{-\tau \wedge \sigma_n} W(X_{\tau \wedge \sigma_n}, S_{\tau \wedge \sigma_n})
\]
But from what was said before
\[
E_{x,s}e^{-\tau \wedge \sigma_n} W(X_{\tau \wedge \sigma_n}, S_{\tau \wedge \sigma_n}) \leq W(x, s),
\]
and letting \( n \to \infty \), by Fatou’s Lemma \( E_{x,s}e^{-\tau} W(X_\tau, S_\tau) \leq W(x, s) \), so the candidate value function is “super harmonic”. Moreover, taking the supremum over all stopping times shows that
\[
W_* (x, s) \leq W(x, s).
\]
To show the reverse inequality, consider \( E_{x,s}e^{-\tau_*} \left( X_{\tau_*} - \kappa V(X_{\tau_*}, S_{\tau_*}) \right) \), where \( \tau_* = \inf \{ t > 0 : X_t \geq h(S_u) \} \) is the optimal stopping time. By definition of the stopping time, one clearly has that this is equal to \( E_{x,s}e^{-\tau_*} (h(s) - \kappa V(h(s), s)) \). Let \( w(x, s) \equiv E_{x,s}e^{-\tau_*} \). It is a well known fact that \( w(x, s) \) itself satisfies \( \mathbb{L}_X w(x, s) = rw(x, s) \). Moreover, for \( x = h(s) \), \( \tau_* = 0 \), thus \( w(x, s) (x - \kappa V(x, s)) \) satisfies the value matching condition and the smooth pasting condition. Consequently
\[
W(x, s) = w(x, s)(x - \kappa V(x, s)) = E_{x,s}e^{-\tau_*} \left( X_{\tau_*} - \kappa V(X_{\tau_*}, S_{\tau_*}) \right) \leq W_* (x, s).
\]
This concludes the verification step.